

Lecture Notes 9.

MA 722

Real Polynomial Optimization II.

Goal

This is a continuation of Lecture Notes 8. Let us repeat the three topics of these notes. We will consider 2. and 3. and its relaxations in this lecture note.

- 1. Positive definite polynomials** Let $f \in \mathbb{R}[x_1, \dots, x_n]$. Decide if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
- 2. Unconstrained polynomial optimization** Let $f \in \mathbb{R}[x_1, \dots, x_n]$. Find $p^* = \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$.
- 3. Polynomial optimization over semi-algebraic sets** Let $f, g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]$ and define $K = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_r(\mathbf{x}) \geq 0\}$. Find $p_K^* = \min_{\mathbf{x} \in K} f(\mathbf{x})$.

Papers [3], [1] and [2] are used to write these lecture notes.

2 Unconstrained polynomial optimization

Let $f \in \mathbb{R}[x_1, \dots, x_n]$. The unconstrained polynomial optimization problem is to find

$$f^* := \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}). \quad (\text{Global-Opt})$$

First we discussed the analogue of the sum of squares relaxation discussed in Lecture Notes 8.

2.1 Sum of squares relaxation

The main idea of the relaxation method is to find the largest possible number $p^* \in \mathbb{R}$ such that $f - p^*$ is a sum of squares, which problem can be solved using semidefinite optimization methods, as discussed below. If the non-negative polynomial $f - f^*$ happens to be a sum of squares, then clearly $f^* = p^*$. Otherwise we have $f^* \geq p^*$. The following definition gives the semidefinite formulation.

Definition 2.1. Let $f = \sum_{|\alpha| \leq 2d} f_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, \dots, x_n]$. Without loss of generality we can assume that the constant term f_0 of f is zero, since $\min_{\mathbf{x} \in \mathbb{R}^n} (f(\mathbf{x}) - f_0) = p^* - f_0$.

The *primal semidefinite optimization problem* is to find :

$$\begin{cases} p^* := \sup \left(-B_0 \bullet Q : Q \in \mathbb{R}_S^{\binom{d+n}{n} \times \binom{d+n}{n}} \right) \\ \text{s.t.} \quad B_\alpha \bullet Q = f_\alpha \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq 2d, \alpha \neq \mathbf{0}, \\ \quad \quad Q \succeq 0, \end{cases} \quad (\text{Pr-Opt})$$

where $B_\alpha \in \mathbb{R}_S^{\binom{d+n}{n} \times \binom{d+n}{n}}$ has 1 in entries that appear in the coefficients of $\mathbf{x}^\alpha$ in the polynomial $\mathbf{X} \cdot Q \cdot \mathbf{X}^T$ with $\mathbf{X} = [\mathbf{x}^\gamma : |\gamma| \leq d]^T$, and 0 otherwise. Note that $B_0 \bullet Q$ is the entry of Q corresponding to the constant term.

The corresponding *dual semidefinite optimization problem* is to find

$$\begin{cases} \bar{p}^* := \inf \left(\sum_{|\alpha| \leq 2d} f_\alpha y_\alpha : \mathbf{y} = [y_\alpha : |\alpha| \leq 2d] \in \mathbb{R}^{\binom{2d+n}{n}} \right) \\ \text{s. t.} \quad y_0 = 1 \text{ and } M_d(\mathbf{y}) \succeq 0, \end{cases} \quad (\text{Du-Opt})$$

where $M_d(\mathbf{y})$ denotes the degree d moment matrix, as in Definition 1.7 in Lecture Notes 8.

The following theorem is a consequence of the Farkas Lemma for SDP's and the fact that there is always a moment matrix such that $M_d(\mathbf{y}) \succ 0$:

Theorem 2.2. *There is no duality gap, i.e. $p^* = \bar{p}^*$. Furthermore, if $\bar{p}^* > \infty$ then (Pr-Opt) has a solution.*

We have the following formulation to find f^* if $f - f^*$ is a sum of squares:

Theorem 2.3. Let $f = \sum_{|\alpha| \leq 2d} f_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, \dots, x_n]$ of total degree $2d$ and assume that $f_0 = 0$. If the nonnegative polynomial $f - f^*$ is a sum of squares then the problem (Global-Opt) is equivalent to the semidefinite optimization problems (Pr-Opt) and (Du-Opt), and $f^* = p^* = \bar{p}^*$. Furthermore, if $\mathbf{x}^* \in \mathbb{R}^n$ such that $f^* = f(\mathbf{x}^*)$, then

$$\mathbf{y}^* = [1, x_1^*, x_2^*, \dots, (x_1^*)^2, x_1^* x_2^*, \dots, (x_n^*)^{2d}]$$

is a minimizer for (Du-Opt).

The natural question that arises is that how can we prove that the optimum p^* that we found with semidefinite programming is the global optimum f^* of f over $\mathbb{R}^n$. The following theorem, which we do not prove here, is a necessary condition for $p^* = f^*$. The proof uses the theory of *flat extensions of moment matrices*.

Theorem 2.4. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be of degree $2d$ with zero constant term, and suppose that the optimal solution $\bar{p}^*$ of (Du-Opt) is attained at $\mathbf{y}^* \in \mathbb{R}^{\binom{2d+n}{n}}$. If

$$\text{rank } M_{d-1}(\mathbf{y}^*) = \text{rank } M_d(\mathbf{y}^*)$$

then $f^* = \bar{p}^* = p^*$. Furthermore, if $\text{rank } M_{d-1}(\mathbf{y}^*) = \text{rank } M_d(\mathbf{y}^*) = r$ then there are r global minimizers.

Example 2.5: Continuing Example 1.3 from Lecture Notes 8, let

$$f(x) = 2x^4 + 2x^3 - x^2 \in \mathbb{R}[x]$$

with zero constant term. We can write $f - f^*$ using a symmetric matrix $Q \in \mathbb{R}_S^{3 \times 3}$

$$\begin{aligned} f(x) - f^* &= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \cdot \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\ &= q_{33}x^4 + 2q_{23}x^3 + (q_{22} + 2q_{13})x^2 + 2q_{12}x + q_{11}. \end{aligned}$$

Comparing coefficients we get the following linear equations for the entries of Q :

$$\{q_{33} = 2, \quad 2q_{23} = 2, \quad q_{22} + 2q_{13} = -1, \quad 2q_{12} = 0, \quad q_{11} = -f^*\}. \quad (1)$$

So the primal semidefinite optimization problem is given by

$$\begin{cases} \sup_{Q \in \mathbb{R}^{3 \times 3}} (-q_{11}) \\ q_{33} = 2, \quad 2q_{23} = 2, \quad q_{22} + 2q_{13} = -1, \quad 2q_{12} = 0 \\ Q \succeq 0. \end{cases}$$

The corresponding dual problem is

$$\begin{cases} \inf_{\mathbf{y} \in \mathbb{R}^5} (2y_4 + 2y_3 - y_2) \\ \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0 \\ y_0 = 1. \end{cases}$$

By solving the dual problem we get that the minimum is -1 and $\mathbf{y}^* = [1, -1, 1, -1, 1]$ is a minimizer. To use Theorem 2.4 to test whether -1 is the global optimum of f we compare

$$\text{rank } M_2(\mathbf{y}^*) = \text{rank} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = 1 = \text{rank } M_1(\mathbf{y}^*) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

therefore

$$\min_{x \in \mathbb{R}} f(x) = -1,$$

and the minimum is taken at $x = -1$ (note that in the univariate case $f - f^*$ is always a sum of squares). The SOS decomposition for $f + 1$ is given by the optimal solution for the primal problem:

$$Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} = L \cdot L^T, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

2.2 Lasserre's relaxation

In the general case, when $f - f^*$ is not a sum of square, we can apply several approaches. One natural approach is to try to find the largest $p_D^* \in \mathbb{R}$ such that a degree D Hilbert-Artin representation for $f - p_D^*$ exists (see Section 1.3 in Lecture Notes 8 for the Hilbert-Artin representation). By gradually increasing D one may prove that we converge to f^* from below.

In this section we give a slightly different approach, which is in the paper [1]. This approach naturally generalizes to *constrained polynomial optimization* problems that we will discuss in the next section. The main idea is to constrain the problem to a compact ball of radius a . Define the quadratic polynomial

$$\theta_a(\mathbf{x}) := a^2 - \|\mathbf{x}\|_2^2,$$

the compact ball $K_a := \{\mathbf{x} \in \mathbb{R}^n : \theta_a(\mathbf{x}) \geq 0\}$, and

$$f_a^* := \min_{\mathbf{x} \in K_a} f(\mathbf{x}).$$

Clearly, if we know a priori that the 2-norm of the global minimizer $\mathbf{x}^*$ is bounded by a then $f^* = \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) = f_a^* = \min_{\mathbf{x} \in K_a} f(\mathbf{x})$. The following result will allow us to formulate the computation of f_a^* as a semidefinite optimization problem:

Theorem 2.6. *Let $f \in \mathbb{R}[x_1, \dots, x_n]$, and assume that $f(\mathbf{x})$ is **strictly positive** for all $\mathbf{x} \in K_a$. Then there exists $p, q \in \mathbb{R}[x_1, \dots, x_n]$, both are sums of squares polynomials, such that*

$$f(\mathbf{x}) = p(\mathbf{x}) + q(\mathbf{x})\theta_a(\mathbf{x}).$$

The degrees of p and q can be higher than the degree of f .

Note that if the degree of p is $2D$ then the degree of q is $2(D-1)$. We get the following semidefinite optimization problem for a fixed D :

Definition 2.7. Let $f = \sum_{|\alpha| \leq 2d} f_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, \dots, x_n]$, and assume that the constant term f_0 of f is zero. Fix $D \geq d$, and $a \in \mathbb{R}$. The *primal semidefinite optimization problem* is to find:

$$\left\{ \begin{array}{ll} p_{a,D}^* & := \sup \left(-P_0 - a^2 Q_0 : P \in \mathbb{R}_S^{\binom{D+n}{n} \times \binom{D+n}{n}}, Q \in \mathbb{R}_S^{\binom{D-1+n}{n} \times \binom{D-1+n}{n}} \right) \\ \text{s.t.} & B_\alpha \bullet P + C_\alpha \bullet Q = f_\alpha \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq 2d, \alpha \neq \mathbf{0}, \\ & B_\alpha \bullet P + C_\alpha \bullet Q = 0 \quad \forall \alpha \in \mathbb{N}^n, |\alpha| > 2d, \\ & P \succeq 0, Q \succeq 0. \end{array} \right.$$

(Primal-Degree-D)

Here for $\alpha \neq 0$ the symmetric matrix $B_\alpha \in \mathbb{R}_S^{\binom{D+n}{n} \times \binom{D+n}{n}}$ is the same as above, and $C_\alpha \in \mathbb{R}_S^{\binom{D-1+n}{n} \times \binom{D-1+n}{n}}$ represent the coefficients of $\mathbf{x}^\alpha$ in the polynomial $\theta_a(\mathbf{x}) \cdot \mathbf{X} \cdot Q \cdot \mathbf{X}^T$ with $\mathbf{X} = [\mathbf{x}^\gamma : |\gamma| \leq D-1]^T$. Also, P_0 and Q_0 denotes the entries corresponding to the constant term in P and Q , respectively.

The corresponding *dual semidefinite optimization problem* is to find:

$$\begin{cases} \bar{p}_{a,D}^* := \inf \left(\sum_{|\alpha| \leq 2d} f_\alpha y_\alpha : \mathbf{y} = [y_\alpha : |\alpha| \leq 2D] \in \mathbb{R}^{\binom{2D+n}{n}} \right) \\ \text{s.t.} & y_0 = 1, \\ & M_D(\mathbf{y}) \succeq 0, \\ & M_{D-1}(\theta_a \mathbf{y}) \succeq 0. \end{cases} \quad (\text{Dual-Degree-D})$$

where $M_D(\mathbf{y})$ denotes the degree D moment matrix as above and $M_{D-1}(\theta_a \mathbf{y})$ is the shifted moment matrix defined in Definition 1.11 in Lecture Notes 8.

The following theorem is due to Lasserre in [1]:

Theorem 2.8. *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree $2d$, $f^* = \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$ and assume that $\|\mathbf{x}^*\| < a$ for the global minimizer. Then*

(a) *as $D \rightarrow \infty$*

$$\bar{p}_{a,D} \uparrow f^*.$$

(b) *$\bar{p}_{a,D} = f^*$ if and only if*

$$f(\mathbf{x}) - f^* = \sum_{i=1}^{r_1} p_i(\mathbf{x})^2 + \theta(\mathbf{x}) \sum_{i=1}^{r_2} q_i(\mathbf{x})^2 \quad (2)$$

for some $p_i \in \mathbb{R}[\mathbf{x}]$ of degree D and $q_i \in \mathbb{R}[\mathbf{x}]$ of degree $D-1$. In addition, for the global minimizer $\mathbf{x}^ \in \mathbb{R}^n$ of f , the vector*

$$\mathbf{y}^* = [(\mathbf{x}^*)^\alpha : |\alpha| \leq 2D] \in \mathbb{R}^{\binom{2D+n}{n}}$$

is a minimizer for the dual problem in degree D .

Proof. (a) From $\mathbf{x}^* \in K_a$ we have for $\mathbf{y}^* = [(\mathbf{x}^*)^\alpha : |\alpha| \leq 2D]$ that

$$M_D(\mathbf{y}^*) \succeq 0 \text{ and } M_{D-1}(\theta_a \mathbf{y}^*) \succeq 0.$$

Therefore $\mathbf{y}^*$ is a feasible solution for the dual for any D , thus f^* must be greater or equal than the infimum $\bar{p}_{a,D}^*$ of the dual for any D .

Now fix $\varepsilon > 0$. Since $f(\mathbf{x}) - (f^* - \varepsilon)$ is strictly positive on K_a , by Theorem 2.6 there exists D_0 such that

$$f(\mathbf{x}) - f^* + \varepsilon = \sum_{i=1}^{r_1} p_i(\mathbf{x})^2 + \theta(\mathbf{x}) \sum_{i=1}^{r_2} q_i(\mathbf{x})^2$$

for some $p_i \in \mathbb{R}[\mathbf{x}]$ of degree D_0 and $q_i \in \mathbb{R}[\mathbf{x}]$ of degree $D_0 - 1$. This shows that the primal problem of degree D_0 is solvable (as well as for any $D \geq D_0$). The weak duality implies that $\bar{p}_{a,D_0}^* \geq p_{a,D_0}^*$, therefore

$$f^* - \varepsilon \leq p_{a,D_0}^* \leq \bar{p}_{a,D_0}^* \leq f^*.$$

The claim that $\bar{p}_{a,D}$ converges from below to f^* follows from the fact that $\bar{p}_{a,D} \geq \bar{p}_{a,D'}$ whenever $D \geq D'$.

(b) " $\Rightarrow$ ": If $\bar{p}_{a,D} = f^*$ the clearly $\mathbf{y}^* = [(\mathbf{x}^*)^\alpha : |\alpha| \leq 2D]$ is a minimizer for the dual at degree D . One can prove that there is no duality gap between $\bar{p}_{a,D}$ and $p_{a,D}$ by constructing $\mathbf{y}$ such that $M_D(\mathbf{y}), M_{D-1}(\theta_s \mathbf{y}) \succ 0$, and using the Farkas Lemma for SDP's. Therefore the primal problem is also solvable with its maximum equal to f^* , and the maximizers P^*, Q^* give the desired decomposition of $f - f^*$ as in (2).

" $\Leftarrow$ ": If a decomposition of $f - f^*$ as in (2) exists then that gives a feasible solution for the primal problem at degree D , thus

$$f^* \leq p_{a,D}^*.$$

But f^* is also an upper bound for $p_{a,D}^*$ for all D since for all $p > f^*$ $f(\mathbf{x}) - p$ must have negative values on K_a . Using again that there is no duality gap we have that $f^* = p_{a,D}^* = \bar{p}_{a,D}^*$. $\square$

3 Real optimization over semi-algebraic sets

3.1 Infeasibility certificates and the Positivstellensatz

First we recall some well-known certificates of infeasibility for different computational problems, giving a context for the statements in the Positivstellensatz.

Range-kernel: Let $A \in F^{m \times n}$, $\mathbf{b} \in F^m$ for some field F . Then

$$\begin{aligned} & A\mathbf{x} = \mathbf{b} \text{ is infeasible in } F^n \\ \Leftrightarrow & \exists \mathbf{y} \in F^m \text{ s. t. } A^T \mathbf{y} = 0, \mathbf{b}^T \mathbf{y} = 1. \end{aligned}$$

Hilbert's Nullstellensatz: For $h_1, \dots, h_m \in \mathbb{C}[x_1, \dots, x_m]$

$$\begin{aligned} & h_1(\mathbf{x}) = \dots = h_m(\mathbf{x}) = 0 \text{ is infeasible in } \mathbb{C}^n \\ \Leftrightarrow & 1 \in \text{Ideal}(h_1, \dots, h_m). \end{aligned}$$

Farkas Lemma: Given $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $C \in \mathbb{R}^{l \times n}$, $\mathbf{d} \in \mathbb{R}^l$.

$$\begin{aligned} & \begin{cases} A\mathbf{x} = \mathbf{b} \\ C\mathbf{x} \geq \mathbf{d} \end{cases} \text{ is infeasible in } \mathbb{R}^n \\ \Leftrightarrow & \exists \mathbf{y} \in \mathbb{R}^m, \mathbf{z} \geq 0 \in \mathbb{R}^l \begin{cases} A^T \mathbf{y} + C^T \mathbf{z} = 0 \\ \mathbf{b}^T \mathbf{y} + \mathbf{d}^T \mathbf{z} = -1 \\ \mathbf{z} \geq 0. \end{cases} \end{aligned}$$

Before we describe the statement of the Positivstellensatz we need some definitions.

Definition 3.1. Let $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$. The *cone* $P = P(f_1, \dots, f_m)$ generated by $f_1, \dots, f_m$ is the smallest subset of $\mathbb{R}[x_1, \dots, x_n]$ containing

1. $f_1, \dots, f_m$,
2. a^2 for all $a \in \mathbb{R}[x_1, \dots, x_n]$,
3. $a + b$ for all $a, b \in P$,
4. ab for all $a, b \in P$.

Note that $P(\emptyset)$ is the set of polynomials which are sum of squares. Also note that

$$P(f_1, \dots, f_m) = \{p_0 + p_1 f_1 + p_{1,2} f_1 f_2 + \dots + p_{i_1, \dots, i_t} f_{i_1} \dots f_{i_t} \mid \forall p_{j_1, \dots, j_u} \in P(\emptyset)\}$$

Let $g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]$. The *multiplicative monoid* $M(g_1, \dots, g_r)$ generated by $g_1, \dots, g_r$ is the set of finite products

$$M(g_1, \dots, g_r) = \{g_{i_1} \cdots g_{i_t} : t \geq 0, 1 \leq i_1 \leq \dots \leq i_t \leq r\}.$$

Note that $M(\emptyset) = \{1\}$.

Next we state the Positivstellensatz, which gives a certificate for the emptiness of a real semi-algebraic set.

Theorem 3.2 (Positivstellensatz). *Let $f_1, \dots, f_m, g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{R}[x_1, \dots, x_n]$. Then*

$$\begin{cases} f_i(\mathbf{x}) \geq 0 & i = 1, \dots, m \\ g_i(\mathbf{x}) \neq 0 & i = 1, \dots, r \\ h_i(\mathbf{x}) = 0 & i = 1, \dots, s \end{cases} \text{ is infeasible in } \mathbb{R}^n$$

$$\Leftrightarrow \exists f \in P(f_1, \dots, f_m), g \in M(g_1, \dots, g_r), h \in \text{Ideal}(h_1, \dots, h_s)$$

$$\text{s.t. } f + g^2 + h = 0.$$

Partial proof. We only prove the easier “ $\Leftarrow$ ” direction. Assume we have $f \in P(f_1, \dots, f_m)$, $g \in M(g_1, \dots, g_r)$, $h \in \text{Ideal}(h_1, \dots, h_s)$ such that $f + g^2 + h = 0$. Suppose $\mathbf{x}_0 \in \{\mathbf{x} \in \mathbb{R}^n : \forall i \ f_i(\mathbf{x}) \geq 0, g_i(\mathbf{x}) \neq 0, h_i(\mathbf{x}) = 0\}$, i.e. it is not empty. But then it is easy to see that $f(\mathbf{x}_0) + g^2(\mathbf{x}_0) + h(\mathbf{x}_0) > 0$, a contradiction. $\square$

For any fixed degree D , one can test the existence of a certificate (f, g, h) of degree at most D using a semidefinite feasibility program: Given $f_1, \dots, f_m, g_1, \dots, g_r, h_1, \dots, h_s \in \mathbb{R}[x_1, \dots, x_n]$ we define f , g and h as follows:

- If $r = 0$ define $g := 1$. If $r \geq 1$ and some of the g_i has positive degrees then $g := \prod_{i=1}^r g_i^m$ for a maximal m such that $\deg g \leq D$.
- Define unknown polynomials $p_{j_1, \dots, j_u}$ such that $f := p_0 + p_1 f_1 + p_{1,2} f_1 f_2 + \dots + p_{i_1, \dots, i_t} f_{i_1} \cdots f_{i_t}$ has degree D and each $p_{j_1, \dots, j_u}$ is a sum of squares. This can be achieved by imposing positive semi-definiteness on matrices of appropriate sizes.
- Define the unknown polynomials $q_1, \dots, q_s$ such that $h = q_1 h_1 + \dots + q_s h_s$ has degree D .

- The linear constraints for the coefficients of $p_{j_1, \dots, j_u}$ and q_j comes from the coefficients of $f + g^2 + h$ being all 0.

Clearly, for a fixed D the polynomials f, g, h defined above are solutions of a semidefinite feasibility problem, but in the most general case the number of unknown variables in this semidefinite program is too high for practical computations. In what follows we discuss a few spacial cases where stronger versions of the Positivstellensatz are valid, which allow more efficient computation.

3.2 Compact semi-algebraic sets

In this subsection we consider compact semi-algebraic sets, and give an extension of the results in Subsection 2.2 of the Lasserre relaxation.

Given $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$, let

$$K = \{\mathbf{x} \in \mathbb{R}^n : f_1(\mathbf{x}) \geq 0, \dots, f_m(\mathbf{x}) \geq 0\}.$$

Assume that K is a compact set. Moreover, to satisfy some technical assumptions we will also assume that the polynomial $\theta_a = \|\mathbf{x}\|_2^2 - a^2$ is among the f_i 's for some $a \in \mathbb{R}$.

Then we have the following stronger version of the Positivstellensatz:

Theorem 3.3. *Let $K = \{\mathbf{x} \in \mathbb{R}^n : f_1(\mathbf{x}) \geq 0, \dots, f_m(\mathbf{x}) \geq 0\}$ be as above. For $f \in \mathbb{R}[x_1, \dots, x_n]$ assume that $F(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$. Then there exists $p_0, p_1, \dots, p_m$ sum of squares polynomials such that*

$$f(\mathbf{x}) = p_0(\mathbf{x}) + \sum_{i=1}^m f(\mathbf{x})p_i(\mathbf{x}).$$

Note that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$ if and only if the set $\{\mathbf{x} \in \mathbb{R}^n : -f(\mathbf{x}) \geq 0, f_1(\mathbf{x}) \geq 0, \dots, f_m(\mathbf{x}) \geq 0\}$ is empty. There for we can apply Theorem 3.2 to certify that $f(\mathbf{x}) > 0$ on K but it gives a weaker version than then Theorem 3.3., involving sum of square multiples of F as well.

Theorem 3.3 can be used to solve optimization problems over compact semi-algebraic sets by solving a sequence of semidefinite optimization problems for increasing values of D . The following formulation is a straightforward generalization of Lasserre's relaxation presented in Subsection 2.2.

Definition 3.4. Given $f_1, \dots, f_m$ such that $K = \{\mathbf{x} \in \mathbb{R}^n : f_1(\mathbf{x}) \geq 0, \dots, f_m(\mathbf{x}) \geq 0\}$ is compact and satisfies the above technical assumptions. Let $f = \sum_{|\alpha| \leq 2d} f_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, \dots, x_n]$ with zero constant term, and define $f_K^* := \min_{\mathbf{x} \in K} f(\mathbf{x})$.

The primal semidefinite optimization problem of degree D is given by

$$\begin{cases} p_{K,D}^* &:= \sup \left(-P_0 - \sum_{i=1}^m f_i(0)P_{i,0} : P_i \in \mathbb{R}_S^{\binom{D+n}{n} \times \binom{D+n}{n}} \right) \\ \text{s.t.} & B_{0,\alpha} \bullet P_0 + \sum_{i=1}^m B_{i,\alpha} \bullet P_i = f_\alpha \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq 2d, \alpha \neq \mathbf{0}, \\ & B_{0,\alpha} \bullet P_0 + \sum_{i=1}^m B_{i,\alpha} \bullet P_i \quad \forall \alpha \in \mathbb{N}^n, |\alpha| > 2d, \\ & P \succeq 0, \dots, P_m 0. \end{cases}$$

Here for $\alpha \neq 0$ the symmetric matrix $B_{0,\alpha} \in \mathbb{R}_S^{\binom{D+n}{n} \times \binom{D+n}{n}}$ correspond to the coefficient of $\mathbf{x}^\alpha$ in $\mathbf{X} \cdot P_0 \cdot \mathbf{X}^T$, while $B_{i,\alpha} \in \mathbb{R}_S^{\binom{D-\deg(f_i)+n}{n} \times \binom{D-\deg(f_i)+n}{n}}$ represent the coefficients of $\mathbf{x}^\alpha$ in the polynomial $f_i(\mathbf{x}) \cdot \mathbf{X} \cdot P_i \cdot \mathbf{X}^T$ for $i = 1, \dots, m$. Here $\mathbf{X} = [\mathbf{x}^\gamma]^T$ is the vector of monomials of appropriate degrees. Also, $P_{i,0}$ denotes the entries corresponding to the constant term in P_i .

The corresponding *dual semidefinite optimization problem* is to find:

$$\begin{cases} \bar{p}_{K,D}^* := & \inf \left(\sum_{|\alpha| \leq 2d} f_\alpha y_\alpha : \mathbf{y} = [y_\alpha : |\alpha| \leq 2D] \in \mathbb{R}^{\binom{2D+n}{n}} \right) \\ \text{s.t.} & y_0 = 1, \\ & M_D(\mathbf{y}) \succeq 0, \\ & M_{D-\deg(f_i)}(f_i \mathbf{y}) \succeq 0 \quad i = 1, \dots, m. \end{cases}$$

where $M_D(\mathbf{y})$ denotes the degree D moment matrix as above and $M_{D-\deg(f_i)}(f_i \mathbf{y})$ is the shifted moment matrix defined in Definition 1.11 in Lecture Notes 8.

Analogously to Theorem 2.8 one can prove that $p_{K,D}^*$ converges to f_K^* from below as D approaches infinity.

3.3 Finite sets

Next we discuss the case when K is a finite algebraic set. We will prove that positive polynomials over finite sets are always sums of squares, modulo the defining equations of the finite set.

Let $h_1, \dots, h_s \in \mathbb{R}[x_1, \dots, x_n]$ such $V_{\mathbb{C}}(h_1, \dots, h_s) \subset \mathbb{C}^n$ is finite, i.e. $h_1, \dots, h_s$ has finitely many roots in $\mathbb{C}^n$. Assume further that $I = \text{Ideal}(h_1, \dots, h_s)$ is a radical ideal. Let

$$K = V_{\mathbb{C}}(h_1, \dots, h_s) \cap \mathbb{R}^n.$$

The next theorem states that positive polynomials are sum of squares modulo the ideal I .

Theorem 3.5. *Let $h_1, \dots, h_s \in \mathbb{R}[x_1, \dots, x_n]$, and K as above. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$. Then there exists a sum of square polynomial p and polynomials $q_1, \dots, q_s$ such that*

$$f(\mathbf{x}) = p(\mathbf{x}) + \sum_{i=1}^s q_i(\mathbf{x})h_i(\mathbf{x}).$$

Proof. Let $V_{\mathbb{C}}(h_1, \dots, h_s) = \{\mathbf{z}_1, \dots, \mathbf{z}_k\} \subset \mathbb{C}^n$, and assume that the first t of them are the real roots. Since I is radical, the factor algebra $A := \mathbb{C}[x_1, \dots, x_n]/I$ has dimension is k . Let $B = \{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_k}\}$ be a normal set for A . Then the Vandermonde matrix

$$V := [\mathbf{z}_j^{\alpha_i}]_{i,j=1}^k \tag{3}$$

is non-singular. For $i = 1, \dots, k$ let $l_i(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$ be the Lagrange basis polynomials with support B , i.e. such that $l_i(\mathbf{z}_j) = \delta_{i,j}$ (note that their coefficients corresponding to the rows of V^{-1}). Then for each $i = 1, \dots, t$ when $\mathbf{z}_i \in \mathbb{R}^n$ we have $l_i \in \mathbb{R}[x_1, \dots, x_n]$ since its complex roots pair up in complex conjugates. Thus for $i = 1, \dots, t$ we can define

$$p_i(\mathbf{x}) = f(\mathbf{z}_i)l_i^2(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n].$$

The $k - t$ complex roots pair up in complex conjugates of each other, and we can assume that the pairs are consecutively numbered. For each such conjugate pairs $\mathbf{z}_j = \bar{\mathbf{z}}_{j+1}$ ($j > t$) we define a polynomial

$$p_{\frac{j+1+t}{2}} := \left(\sqrt{f(\mathbf{z}_j)}l_j(\mathbf{x}) + \sqrt{f(\mathbf{z}_{j+1})}l_{j+1}(\mathbf{x}) \right)^2.$$

Again, $p_{\frac{j+1+t}{2}} \in \mathbb{R}[x_1, \dots, x_n]$, since $\sqrt{f(\mathbf{z}_j)}l_j(\mathbf{x})$ is the complex conjugate of $\sqrt{f(\mathbf{z}_{j+1})}l_{j+1}(\mathbf{x})$. Let

$$p(\mathbf{x}) := \sum_{i=1}^{(t+n)/2} p_i(\mathbf{x}). \quad (4)$$

Since $f(\mathbf{z}_i) > 0$ for all $\mathbf{z}_i \in K$ we have that p is a sum of squares. Furthermore, for all $i = 1, \dots, k$ we have that

$$p(\mathbf{z}_i) = f(\mathbf{z}_i),$$

therefore, since I is radical, $f = p \pmod I$. Thus there exists $q_1, \dots, q_s$ such that

$$f(\mathbf{x}) = p(\mathbf{x}) + \sum_{i=1}^s q_i(\mathbf{x})h_i(\mathbf{x}).$$

Lastly, $q_i \in \mathbb{R}[x_1, \dots, x_n]$ since its coefficients are solutions of a linear system with real coefficients. $\square$

Note that if all common roots of $h_1, \dots, h_s$ are real, then the sum of square polynomial p defined in (4) is equal to

$$p(\mathbf{x}) = X^T \cdot (V^{-1})^T \cdot \text{diag}(f(\mathbf{z}_1), \dots, f(\mathbf{z}_k)) \cdot V^{-1} \cdot X,$$

where V is the Vandermonde matrix defined in (3), and $X = [\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_k}]^T$. Recall that the *Hermite matrix* of g modulo I is defined as

$$H_g = V^T \text{diag}(g(\mathbf{z}_1), \dots, g(\mathbf{z}_k))V$$

for any $g \in \mathbb{R}[x_1, \dots, x_n]$. Thus,

$$(V^{-1})^T \text{diag}(f(\mathbf{z}_1), \dots, f(\mathbf{z}_k))V^{-1} = H_{1/f}^{-1}.$$

This implies that in the case when $h_1, \dots, h_s$ has only real roots, we can compute $p(x)$ without computing the roots $\mathbf{z}_1, \dots, \mathbf{z}_k$, using *traces*, as was discussed in an earlier class note.

Example 3.6: Let

$$f = x^2 + y^2 - z^2 + 1, \quad \begin{cases} h_1 = yx - z \\ h_2 = zy - x \\ h_3 = zx - y. \end{cases}$$

We will show that h_1, h_2, h_3 has only real roots, f is positive on these roots, and give a sum of square decomposition of f modulo $I = \langle h_1, h_2, h_3 \rangle$. We do this without computing the common roots of I , but rather by computing the inverse of the Hermite matrix $H_{1/f}^{-1}$.

First we compute a Gröbner basis for I with respect to the graded lexicographic ordering $x > y > z$:

$$G = [zy - x, -z^2 + y^2, zx - y, yx - z, -z^2 + x^2, -z + z^3]$$

Thus the normal set of I w.r.t. this ordering is $B = [1, z, y, x, z^2]$. Using the Gröbner bases we can compute $g = 1/f \mod I$ as follows: If

$$g = 1/f = g_0 + g_1x + g_2y + g_3z + g_4z^2$$

then we get that fg reduced by G is

$$\text{Reduce}(fg, G) = g_0 + 2g_3z + 2g_1x + 2g_2y + (2g_4 + g_0)z^2$$

which has to be 1. Comparing coefficients we get that

$$g = 1 - 1/2 z^2.$$

Now the Hermite matrix of g w.r.t. the normal set $B = [1, z, y, x, z^2]$ is

$$H_g = \begin{bmatrix} \text{Tr}(M_g) & \text{Tr}(M_{zg}) & \text{Tr}(M_{yg}) & \text{Tr}(M_{xg}) & \text{Tr}(M_{z^2g}) \\ \text{Tr}(M_{zg}) & \text{Tr}(M_{z^2g}) & \text{Tr}(M_{yzg}) & \text{Tr}(M_{z^3g}) & \text{Tr}(M_{z^4g}) \\ \text{Tr}(M_{yg}) & \text{Tr}(M_{yzg}) & \text{Tr}(M_{y^2g}) & \text{Tr}(M_{xyg}) & \text{Tr}(M_{yz^2g}) \\ \text{Tr}(M_{xg}) & \text{Tr}(M_{xzg}) & \text{Tr}(M_{xyg}) & \text{Tr}(M_{x^2g}) & \text{Tr}(M_{xz^2g}) \\ \text{Tr}(M_{z^2g}) & \text{Tr}(M_{z^3g}) & \text{Tr}(M_{z^2yg}) & \text{Tr}(M_{z^2xg}) & \text{Tr}(M_{z^4g}) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix},$$

where $\text{Tr}(M_q)$ is the trace of the multiplication matrix of $q \bmod I$. We can see that H_g is strictly positive definite, which implies that all common roots of I are real, and that f is strictly positive on the roots. We get the sum of square decomposition of f by computing the the Cholesky factorization of H_g^{-1} :

$$H_g^{-1} = L \cdot L^T, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1/2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1/2\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & 1/2\sqrt{2} \end{bmatrix}.$$

Multiplying $H_g^{-1} = L \cdot L^T$, L by the vector $[1, z, y, x, z^2]$ and its transpose from both sides we get that

$$\begin{aligned} p &= 1 - 3/2 z^2 + 1/2 y^2 + 1/2 x^2 + 3/2 z^4 \\ &= (1 - z^2)^2 + (1/\sqrt{2} z)^2 + (1/\sqrt{2} y)^2 + (1/\sqrt{2} x)^2 + (1/\sqrt{2} z^2)^2, \end{aligned}$$

is a sum of squares, and

$$\begin{aligned} f - p &= -\frac{1}{2} (z^2 + y^2 + x^2 - 3z^4) \\ &= -\frac{1}{2} ((3z^3 - z)h_1 + (-3z^2x - x)h_2 + (-3zx - y)h_3), \end{aligned}$$

giving the proof that f is a sum of squares modulo the ideal.

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