# Lecture Notes 8.

MA 722

# Real Polynomial Optimization I.

## Goal

The ultimate goal is to solve the following three problems. Since these problems are hard in general, we will study *relaxations* of them.

1. **Positive definite polynomials** Let $f \in \mathbb{R}[x_1, \ldots, x_n]$. Decide if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

2. **Unconstrained polynomial optimization** Let $f \in \mathbb{R}[x_1, \ldots, x_n]$. Find $p^* = \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$.

3. **Polynomial optimization over semi-algebraic sets** Let $f, g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$ and define $K = \{\mathbf{x} \in \mathbb{R}^n \ : \ g_1(\mathbf{x}) \geq 0, \ldots, g_r(\mathbf{x}) \geq 0\}$. Find $p_K^* = \min_{\mathbf{x} \in K} f(\mathbf{x})$.

Papers [4], [3] and [1] are used to write these lecture notes.

## 1 Positive definite polynomials

First we will discuss a relaxation of the problem of deciding if $f$ is positive definite to a simpler problem, namely, to decide whether $f$ can be expressed as a sum of squares. Then we will study the general case.

**Definition 1.1.** Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ and assume that the total degree of $f$ is $2d$. Then $f$ is a *sum of squares (SOS)* if there exists $q_1, \ldots, q_t \in \mathbb{R}[x_1, \ldots, x_n]$ of degrees at most $d$ such that $f = \sum_{i=1}^{t} q_i^2$.

Clearly SOS polynomials are positive definite. However the following example due to Motzkin shows that not all positive definite polynomials are SOS.

**Example 1.2:** The Motzkin polynomial is

$$M(x,y,z) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$$

is positive definite, but not a sum of squares. To see that $M(x,y,z)$ is positive definite we use the inequality between the arithmetic and geometric means:

$$\frac{x^4y^2 + x^2y^4 + 1}{3} \geq \sqrt[3]{x^4y^2 \cdot x^2y^4 \cdot 1} = x^2y^2.$$

To prove that $M(x,y,z)$ is not an SOS one can use techniques described later in these lecture notes, by constructing an exact solution for the *dual* semidefinite problem (see the actual certificate in [1, Example 5.1]).

The following simple example illustrates that finding an SOS decomposition for $f$ can be reduced to finding a positive semi-definite symmetric matrix satisfying certain linear constrains.

**Example 1.3:** Let

$$f(x) = 2x^4 + 2x^3 - x^2 + 5 \in \mathbb{R}[x].$$

We can write $f$ using a symmetric matrix $Q \in \mathbb{R}_S^{3\times 3}$ (we will use the notation $\mathbb{R}_S^{N\times N}$ for the vector space of symmetric $N \times N$ real matrices):

$$\begin{aligned} f(x) &= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \cdot \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\ &= q_{33}x^4 + 2q_{23}x^3 + (q_{22} + 2q_{13})x^2 + 2q_{12}x + q_{11}. \end{aligned}$$

Comparing coefficients we get the following linear equations for the entries of $Q$:

$$\{q_{33} = 2, \quad 2q_{23} = 2, \quad q_{22} + 2q_{13} = -1, \quad 2q_{12} = 0, \quad q_{11} = 5\}. \tag{1}$$

Any positive semi-definite matrix $Q$ that satisfies these linear constrains gives an SOS decomposition for $f$. For example,

$$Q = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 5 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

satisfies the the constrains in (1) and is positive semi-definite:

$$Q = L^T \cdot L \;\text{ with }\; L = \frac{1}{\sqrt{2}} \begin{bmatrix} -3 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

which gives the SOS decomposition

$$\begin{aligned} f(x) &= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \cdot L^T \cdot L \cdot \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\ &= \frac{1}{2}(2x^2 + x - 3)^2 + \frac{1}{2}(3x+1)^2. \end{aligned}$$

The next subsection gives a brief introduction to semidefinite programming, which we will use to solve the SOS problem.

## 1.1 Semi-definite Programming

There are many different notation to define semi-definite programs, here we adopt the following the one using the trace inner product of matrices.

**Definition 1.4.** Denote by $\mathbb{R}_S^{N\times N}$ the space of $N \times N$ real symmetric matrices, and for $X \in \mathbb{R}_S^{N\times N}$ denote by $X \succeq 0$ if $X$ is positive semi-definite. First we define the following inner product on $\mathbb{R}_S^{N\times N} \times \mathbb{R}_S^{N\times N}$. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}_S^{N\times N}$

$$A \bullet B := tr(A \cdot B) = \sum_{i,j=1}^{N} a_{ij} b_{ij},$$

which is simply the scalar product of the "flattenings" of the matrices.

Next we define different versions of the semi-definite programming problem:

**Semi-definite feasibility** Given $A_i \in \mathbb{R}_S^{N\times N}$ for $i = 1, \dots M$ and $\mathbf{b} = [b_1, \dots, b_M]^T \in \mathbb{R}^M$. The *primal semidefinite feasibility* problem is to find $Q \in \mathbb{R}_S^{N\times N}$ subject to

$$\begin{cases} A_i \bullet Q = b_i \quad i = 1, \dots, M, \\ Q \succeq 0. \end{cases} \tag{Pr-Feas}$$

The *dual semidefinite feasibility* problem is to find $\mathbf{w} = [w_1, \ldots, w_M]^T \in \mathbb{R}^M$ such that

$$\begin{cases} \mathbf{b}^T\mathbf{w} < 0 \\ \sum_{i=1}^M w_i A_i \succeq 0. \end{cases} \tag{Du-Feas}$$

**Semi-definite optimization** Given $C \in \mathbb{R}_S^{N\times N}$, $A_i \in \mathbb{R}_S^{N\times N}$ for $i = 1, \ldots M$ and $\mathbf{b} = [b_1, \ldots, b_M]^T \in \mathbb{R}^M$. The *primal semi-definite optimization* problem is to find

$$\begin{cases} \sup_{Q\in\mathbb{R}_S^{N\times N}} C \bullet Q \\ A_i \bullet Q = b_i \quad i = 1, \ldots, M, \\ Q \succeq 0. \end{cases} \tag{Pr-SDP}$$

The *dual semidefinite optimization* problem is to find

$$\begin{cases} \inf_{\mathbf{w}\in\mathbb{R}^M} \mathbf{b}^T\mathbf{w} \\ C + \sum_{i=1}^M w_i A_i \succeq 0. \end{cases} \tag{Du-SDP}$$

**Continuation of Example 1.3:** The SOS decomposition for $f(x) = 2x^4 + 2x^3 - x^2 + 5$ is a primal semidefinite feasibility problem, where $N = 3$, $M = 5$, and the equations in (1) can be described using the trace product of symmetric matrices. For example the equation $q_{22} + 2q_{13} = -1$ is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} = -1.$$

So the coefficient matrices of the 5 equations in (1) are

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding dual feasibility problem is to find $\mathbf{w} = [w_1, w_2, w_3, w_4, w_5] \in \mathbb{R}^5$ such that

$$\mathbf{b}^T\mathbf{w} = 2w_1 + 2w_2 - w_3 + 5w_5 < 0$$
$$W := \mathcal{A}^*(\mathbf{w}) = \begin{bmatrix} w_5 & w_4 & w_3 \\ w_4 & w_3 & w_2 \\ w_3 & w_2 & w_1 \end{bmatrix} \succeq 0.$$

Note that $W$ has a Hankel structure, and if $\mathbf{w} = [x^4, x^3, x^2, x, 1]$ then

$$W = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & x^2 \end{bmatrix}.$$

Such matrices in the multivariate case are called *moment matrices*, which we will define below. Also note that

$$tr(Q \cdot W) = q_{33}w_1 + 2q_{23}w_2 + (q_{22} + 2q_{13})w_3 + 2q_{12}w_4 + q_{11}w_5$$

so if $Q$ satisfies the linear equations above then

$$tr(Q \cdot W) = 2w_1 + 2w_2 - w_3 + 5w_5 = \mathbf{b}^T\mathbf{w}.$$

Thus, if $W$ and $Q$ both positive semi-definite matrices, then their product is also positive semi-definite, so its trace is $\geq 0$. This shows that the primal and the dual problems cannot be feasible in the same time.

The classical *Farkas Lemma* for linear programming connect the feasibility of the primal and dual programs. For semi-definite programming an extra condition is needed:

**Theorem 1.5** (Farkas Lemma for SDP)**.** *Let $A_i \in \mathbb{R}_S^{N\times N}$ for $i = 1, \ldots M$ and $\mathbf{b} \in \mathbb{R}^M$. Suppose there exists a vector $\mathbf{w} \in \mathbb{R}^M$ such that*

$$\sum_{i=1}^{M} w_i A_i \succ 0, \tag{2}$$

*i.e. strictly positive definite. Then exactly one of the following is true:*

1. *There exists $Q \in \mathbb{R}_S^{N\times N}$ such that $A_i \bullet Q = b_i$ for $i = 1, \ldots, M$ and $Q \succeq 0$.*

*2. There exists* $\mathbf{w} \in \mathbb{R}^m$ *such that* $\mathbf{b}^t\mathbf{w} < 0$ *and* $\sum_{i=1}^{M} w_i A_i \succeq 0$.

Note that one direction of the Farkas Lemma for SDP's is always true, even without condition (2), namely that at most one of statements are true. The following example demonstrates that the extra condition in (2) is needed to assure that exactly one of the statements is true:

**Example 1.6:** Consider the following Primal Semidefinite Feasibility Problem:

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \succeq 0$$
$$q_{33} - q_{11} = 0,\ -q_2 2 = 0,\ 2q_{12} = -1.$$

This problem is infeasible, since for positive semi-definite matrices $q_{22} = 0$ implies that $q_{12} = 0$.
The corresponding Dual problem is to find $\mathbf{w} = [w_1, w_2, w_3] \in \mathbb{R}^3$ such that

$$W = \begin{bmatrix} -w_1 & w_3 & 0 \\ w_3 & -w_2 & 0 \\ 0 & 0 & w_1 \end{bmatrix} \succeq 0$$
$$-w_3 < 0.$$

This problem is also infeasible, since for any feasible solutions we have $w_1 = 0$, which implies that $w_3 = 0$. Note that in this case there is no $\mathbf{w}$ such that $W \succ 0$, so (2) is not satisfied.

## 1.2 Deciding if $f$ is a sum of squares

In this subsection we formally express the decision problem of whether $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $2d$ is a sum of squares as a Primal semi-definite feasibility problem, state its dual, and prove that condition (2) in the Farkas Lemma for SDP's is always satisfied for SOS problem.

To understand better the structure of dual problem, we need to define moment matrices:

**Definition 1.7.** Let $\mathbf{y} = [y_\alpha \ : \ \alpha \in \mathbb{N}^n |\alpha| \leq 2d] \in \mathbb{R}^{\binom{2d+n}{n}}$, or a vector o indeterminates, where the entries are indexed by exponent vectors of monomials in $n$ variables of degree at most $2d$. The degree $d$ *moment matrix* of $\mathbf{y}$

is a $\binom{d+n}{n} \times \binom{d+n}{n}$ matrix with rows and column corresponding to monomials in $n$ variables of degree at most $d$, and defined as

$$M_d(\mathbf{y}) = [y_{\alpha+\beta}]_{|\alpha|,|\beta|\leq d}\,.$$

**Example 1.8:** If $n = 1$ then $M_d(\mathbf{y})$ is a Hankel matrix, for example

$$M_3(\mathbf{y}) = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \end{bmatrix}.$$

For $n = 2$ and $d = 2$ the rows and columns correspond to the monomials $[1, x_1, x_2, x_1^2, x_1x_2, x_2^2]$, so the moment matrix is

$$M_2(\mathbf{y}) = \begin{bmatrix} y_{0,0} & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\ y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\ y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{2,3} \\ y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4} \end{bmatrix}.$$

The following theorem expresses the decision problem wether $f$ is SOS as a semi-definite feasibility problem.

**Theorem 1.9.** *Let $f = \sum_{|\alpha|\leq 2d} f_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$ of total degree $2d$.*

1. *$f$ is a sum of squares polynomial if and only if the following Primal semi-definite problem is feasible:*

$$\begin{aligned} &Q \in \mathbb{R}_S^{\binom{d+n}{n}\times\binom{d+n}{n}},\ Q \succeq 0 \\ &B_\alpha \bullet Q = f_\alpha \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq 2d, \end{aligned}$$

*where $B_\alpha \in \mathbb{R}_S^{\binom{d+n}{n}\times\binom{d+n}{n}}$ has 1 in entries that appear in the coefficients of $\mathbf{x}^\alpha$ in the polynomial $\mathbf{X} \cdot Q \cdot \mathbf{X}^T$ with $\mathbf{X} = [\mathbf{x}^\gamma : |\gamma| \leq d]^T$, and 0 otherwise.*

2. *$f$ is not a sum of squares polynomial if and only if following Dual semi-definite problem is feasible:*

$$\begin{aligned}
&\mathbf{y} = [y_\alpha \ : \ |\alpha| \leq 2d] \in \mathbb{R}^{\binom{2d+n}{n}} \\
&\sum_{|\alpha| \leq 2d} f_\alpha y_\alpha < 0 \\
&M_d(\mathbf{y}) \succeq 0.
\end{aligned}$$

*Proof.* The theorem follows from the same construction as in Example 1.3 and the Farkas Lemma for SDP's. The only thing we need to show is that condition (2) is always satisfied for moment matrices, i.e. there exist moment matrices for any $n$ and $d$ that are strictly positive definite. This is true, since for any $d$ there exists $\binom{d+n}{n}$ points $\mathbf{z}_1, \ldots, \mathbf{z}_{\binom{d+n}{n}} \in \mathbb{R}^n$ such that the square Vandermonde matrix $V := [\mathbf{z}_i^\alpha]_{i,\alpha}$ (with rows corresponding to the points and columns corresponding to monomials of degree at most $d$) is non-singular. Then $M_d := V^T \cdot V$ is a moment matrix which is strictly positive definite. $\square$

## 1.3 General Hilbert-Artin representation

As we have seen at the beginning of the section, not all positive definite polynomials are SOS. The following theorem gives a more general representation of positive definite polynomials. This representation is called *Hilbert-Artin representation* in [1]. A version of this theorem was proved by Artin in 1927.

**Theorem 1.10.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ and suppose that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Then there exist $p_1, \ldots, p_s, q_1, \ldots, q_t \in \mathbb{R}[x_1, \ldots, x_n]$, not all zero, such that*

$$\left( \sum_{i=1}^{t} q_i^2 \right) f = \sum_{j=1}^{s} p_i^2. \tag{3}$$

*In other words, $f$ is a ratio of two sum of squares polynomials.*

Notice that if $f$ is SOS, then $t = 1$, $q_1 = 1$ will give the Hilbert-Artin representation of $f$. However, if $f$ is not an SOS, we may need higher degree $q_i$ polynomials. For example, the Motzkin polynomial if Example 1.2 has a Hilbert-Artin representation with linear $q_i$'s (see [1]).

Since we do not have an a priori bound for the degree of the polynomials $q_i$, one can increment this degree, set up the corresponding semidefinite feasibility programs as described below, and test whether they are feasible or not.

We use the following notation:

$$e := \max_i \deg q_i, \quad d := \left\lceil e + \frac{\deg f}{2} \right\rceil.$$

Note that $\deg p_i \leq d$ and both sides of (4) has degrees $2d$.

In order to translate the Hilbert-Artin representation to a semi-definite feasibility problem, we need to define the so called *shifted moment matrices:*

**Definition 1.11.** Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ and $e, d$ as above. Let $\mathbf{y} = [y_\alpha : \alpha \in \mathbb{N}^n, |\alpha| \leq 2d]$ be a vector of indeterminates. Then the *shifted moment matrix* of degree $e$ is defined by

$$M_e(f\mathbf{y}) = \left[\mathbf{x}^{\alpha+\beta} f|_{\mathbf{y}}\right]_{|\alpha|,|\beta| \leq e},$$

where we use the notation that for any polynomial $g = \sum_{\alpha \in \mathbb{N}^n} g_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$

$$g|_{\mathbf{y}} := \sum_{\alpha \in \mathbb{N}^n} g_\alpha y_\alpha$$

the linearization of of $g$.

**Example 1.12:** For $n = 2$, $e = 1$ and $f = x_1^2 + x_2^2 - 1$ we have that

$$M_1(f\mathbf{y}) = \begin{bmatrix} y_{2,0} + y_{0,2} - y_{0,0} & y_{3,0} + y_{1,2} - y_{1,0} & y_{2,1} + y_{0,3} - y_{0,1} \\ y_{3,0} + y_{1,2} - y_{1,0} & y_{4,0} + y_{2,2} - y_{2,0} & y_{3,1} + y_{1,3} - y_{1,1} \\ y_{2,1} + y_{0,3} - y_{0,1} & y_{3,1} + y_{1,3} - y_{1,1} & y_{2,2} + y_{0,4} - y_{0,2} \end{bmatrix}.$$

*Now we are ready express the Hilbert-Artin representation for a fixed degree $e$ as a Primal semidefinite feasibility problem, and state its dual as well.*

**Theorem 1.13.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ and fix $e \geq 0$. Let $d = \left\lceil e + \frac{\deg f}{2} \right\rceil$ as above.*

1. *There exist* $p_1, \ldots, p_s, q_1, \ldots, q_t \in \mathbb{R}[x_1, \ldots, x_n]$*, not all zero, such that*

$$\left(\sum_{i=1}^{t} q_i^2\right) f = \sum_{j=1}^{s} p_i^2 \text{ and } \deg q_i \leq e \tag{4}$$

*if and only if the following primal semidefinite problem is feasible:*

$$\begin{aligned}
&P \in \mathbb{R}_S^{\binom{d+n}{n}\times\binom{d+n}{n}}, Q \in \mathbb{R}_S^{\binom{e+n}{n}\times\binom{e+n}{n}},\\
&P \succeq 0,\ Q \succeq 0\\
&\mathrm{coeffs}_{\mathbf{x}}(\mathbf{X}_d P \mathbf{X}_d^T - f(\mathbf{x}) \cdot \mathbf{X}_e Q \mathbf{X}_e^T) = \mathbf{0}\\
&tr(Q) = 1
\end{aligned}$$

*where* $\mathbf{X}_t = [\mathbf{x}^\alpha \ : \ \alpha \in \mathbb{N}^n, |\alpha| \leq t]$ *and* $\mathrm{coeff}_{\mathbf{x}}(q)$ *denotes the vector of coefficients of* $q \in \mathbb{R}[x_1, \ldots, x_n]$*. Note that* $\mathbf{X}_d P \mathbf{X}_d^T - f(\mathbf{x}) \cdot \mathbf{X}_e Q \mathbf{X}_e^T$ *is a polynomial, and its coefficients in* $\mathbf{x}$ *are linear in the entries of* $P$ *and* $Q$*.*

2. $f$ *has no Hilbert-Artin representation with* $\deg q_i \leq e$ *if and only if the following dual semidefinite program is feasible:*

$$\begin{aligned}
&\mathbf{y} = [y_\alpha \ : \ \alpha \in \mathbb{N}^n, |\alpha| \leq 2d] \in \mathbb{R}^{\binom{2d+n}{n}}, t \in \mathbb{R}\\
&M_d(\mathbf{y}) \succeq 0,\ M_e(-f\mathbf{y}) + tI \succeq 0,\\
&t < 0,
\end{aligned}$$

*where* $I$ *is the* $\binom{e+n}{n} \times \binom{e+n}{n}$ *identity matrix.*

*Proof.* The proof is similar to the proof of Theorem 1.9, and it can be found in [1]. A key part is to construct $\mathbf{y}$ such that $M_d(\mathbf{y}) \succ 0,\ M_e(-f\mathbf{y}) + tI \succ 0$, so that we can apply Farkas Lemma. □

**Continuation of Example 1.2:** For the Motzkin polynomial

$$M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$$

to get a certificate that it is not an SOS, one have to find an exact solution for the dual semidefinite program for $e = 0$. Note that $M_0(-f\mathbf{y}) = f|_{\mathbf{y}}$ is a $1 \times 1$ matrix, so $M_0(-f\mathbf{y}) + tI \succeq 0, t < 0$ is equivalent to $f|_{\mathbf{y}} < 0$ which appears in the dual problem in Theorem 1.9.

In [1] they give the following certificate. First note that we can exploit the sparsity of the Motzkin polynomial, and it is sufficient to consider the following monomials $\mathbf{X} = [1, xy, x^2y, xy^2]$ in the primal program, which gives for the dual the unknown vector

$$\mathbf{y} = [y_{0,0}, y_{1,1}, y_{1,2}, y_{2,2}, y_{3,2}, y_{2,3}, y_{4,2}, y_{3,3}, y_{2,4}].$$

Then the corresponding moment matrix is

$$M_{\mathbf{X}}(\mathbf{y}) = \begin{bmatrix} y_{0,0} & y_{1,1} & y_{2,1} & y_{1,2} \\ y_{1,1} & y_{2,2} & y_{3,2} & y_{2,3} \\ y_{2,1} & y_{3,2} & y_{4,2} & y_{3,3} \\ y_{1,2} & y_{2,3} & y_{3,3} & y_{2,4} \end{bmatrix}.$$

Also

$$f|_{\mathbf{y}} = y_{4,2} + y_{2,4} + y_{0,0} - 3y_{2,2}.$$

A feasible solution is

$$\mathbf{y}^* = [y_{0,0} = y_{1,1} = y_{1,2} = 0, y_{2,2} = 1, y_{3,2} = y_{2,3} = y_{4,2} = y_{3,3} = y_{2,4} = 0],$$

i.e.

$$M_{\mathbf{X}}(\mathbf{y}^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0,$$

$$f|_{\mathbf{y}^*} = -3 \cdot 1 < 0.$$

Note that this shows that any polynomial with the same support as the Motzkin polynomial is not an SOS if the coefficient of $x^2y^2$ is negative.

For $e = 1$ we can solve the primal semi-definite feasibility problem in Theorem 1.13. To lower the size of the semi-definite program, one can start to use a subset of the monomials of degree $e = 1$, and in this case $\mathbf{X}_1 = [1, x]$ will give a feasible solution. Again, using the sparsity of $M(x, y)$ it is sufficient to consider the vector of monomials $\mathbf{X}_4 = [1, x, xy, x^2y, xy^2, x^3y, x^2y^2]$. That leads to a $2 \times 2$ matrix for $Q$ and a $7 \times 7$ matrix for $P$. In [2] they give the following soluteion:

$$(1 + x^2) \cdot M(x, y) = (1 - x^2y^2)^2 + (xy - x^3y)^2 + (x - xy^2)^2.$$

# References


[1] *Feng Guo, Erich L. Kaltofen, and Lihong Zhi. Certificates of impossibility of hilbert-artin representations of a given degree for definite polynomials and functions.* `arxiv.org/abs/1203.0253`*, 2012.*

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[3] *Jean B. Lasserre. Global optimization with polynomials and the problem of moments.* SIAM J. Optim.*, 11(3):796–817 (electronic), 2001.*

[4] *Pablo A. Parrilo. Semidefinite programming relaxations for semialgebraic problems.* Math. Program.*, 96(2):293–320, 2003.*