Real Roots of Univariate Polynomials with Real Coefficients

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1 Real Closed Fields

Before we study real solutions of real polynomials, we give an axiomatic treatment of what makes the field of real numbers special. In this section we give a completely algebraic construction of *real closed fields* that allows to prove some of the classic theorems of real analysis, such as Rolle's Theorem or the Intermediate Value Theorem. We mostly follow that book [BPR06].

1.1 Ordered Fields

Definition 1.1. K is an *ordered field* if it is a commutative field with a subset $P \subset K$ such that

1. $0 \notin P$
2. If $a \in K$ then exactly one is true: either $a \in P$ or $-a \in P$ or $a = 0$ (trichotomy)
3. P is closed under addition and multiplication.

Definition 1.2. Let K be a field. A total ordering $\leq$ on K is compatible with the field operations if for all $a, b, c \in K$

1. $a \leq b \Rightarrow a + c \leq b + c$,
2. $a \leq b, c \geq 0 \Rightarrow ac \leq bc$.

Proposition 1.3. *K is an ordered field if and only if it has a total ordering that is compatible with the field operations.*

Proof. If (K, P) is an ordered field, then we can define the total ordering for $a, b \in K$

$$a \leq b \quad \Leftrightarrow \quad b - a \in P \text{ or } b - a = 0.$$

We can see that this ordering is a transitive relation, defines a total order, and it satisfies the two properties of Definition 1.2.
To prove the other direction, if K has a field compatible total order, then we can define

$$P := \{a \in K \ : \ a \geq 0, a \neq 0\}.$$

Then one can see that P satisfies the 3 properties in Definition 1.1. $\square$

Note that in ordered fields we can define

- intervals: closed, open, half-open;

- absolute value function
- sign function: $\mathrm{sign}(a) = \begin{cases} +1 & \text{if } a \in P \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a \neq 0, a \notin P \end{cases}$

The following are small statements about ordered fields:

Proposition 1.4. *Let (K, P) be an ordered field. Then $-1 \notin P$*

Proof. Suppose $-1 \in P$. Since P is closed under multiplication, we have $(-1)^2 = 1 \in P$, and since P is closed under addition, we have $(-1) + 1 = 0 \in P$, a contradiction. $\square$

Corollary 1.5. *Let (K, P) be an ordered field. For all $a \in K$, $a \neq 0$, we have $a^2 \in P$. In particular, there is no $a \in K$ such that $-1 = a^2$.*

Proof. Let $a \in K$ such that $a \neq 0$. Then by trichotomy either $a \in P$ or $-a \in P$. If $a \in P$ then $a^2 \in P$. If $-a \in P$ then $(-a)^2 = a^2 \in P$. $\square$

Proposition 1.6. *Let (K, P) be a n ordered field, and $K' \subset K$ be a subfield. Then $(K', P \cap K')$ is also an ordered field.*

Proof. We show that $P \cap K'$ satisfy the 3 properties of Definition 1.1. To see 1, $p \notin P \cap K'$ since $0 \notin P$. To prove 2, let $a \in K' \subset K$. Then $a \in P$ if and only if $a \in P \cap K'$, which proves the trichotomy. Also, since both K' and P are closed under addition and multiplication, their intersection is as well, proving 3. $\square$

Next we define formally real fields.

Definition 1.7. A field K is a *formally real field* if for all $a_1, \ldots, a_n \in K$ we have

$$\sum_{i=1}^{n} a_i^2 = 0 \quad \Rightarrow \quad a_1 = \cdots = a_n = 0.$$

Corollary 1.8. *The following statements hold:*

1. *K is an ordered field $\Rightarrow$ K is formally real;*
2. *K is formally real $\Leftrightarrow$ -1 is not a sum of squares;*
3. *K is formally real $\Rightarrow$ $\mathrm{char}(K) = 0$.*

Proof. 1. follows from Corollary 1.5.
2. "$\Rightarrow$": If $-1 = \sum_{i=1}^{n} a_i^2$ then $\sum_{i=1}^{n} a_i^2 + 1^2 = 0$, implying that $1 = 0$, a contradiction.
"$\Leftarrow$": Suppose $\sum_{i=1}^{n} a_i^2$ for some $a_i \in K$, and suppose that $a \not= 0$. Then $1 + \sum_{i=2}^{n} \frac{a_i^2}{a_1^2} = 0$, which expresses -1 as a sum of squares.
3. If $\mathrm{char}(K) = p > 0$ then $\sum_{i=1}^{p} 1^2 = 0$ in K, which contradicts to $1 \neq 0$. $\square$

The next definition gives alternative ways to construct ordered fields.

Definition 1.9. Let K be a field. $C \subset K$ is called a *proper cone* if

1. $-1 \notin C$,

2. for all $a \in K$, $a^2 \in K$,

3. C is closed under addition and multiplication.

The next theorem asserts that any proper cone in K can be extended to give P to make K an ordered field.

Theorem 1.10. *Let K a field and $C \subset K$ be a proper cone. Then there exists $P \subset K$ such that $C \subset P \cup \{0\}$ and P satisfies the three conditions of Definition 1.1.*

To prove the theorem, we need the following lemma:

Lemma 1.11. *Let K a field and $C \subset K$ be a proper cone. If $-a \notin C$ then*

$$C[a] := \{x + ay \ : \ x, y, \in C\}$$

is also a proper cone.

Proof of Lemma. If $-1 \in C[a]$ then there exists $x, y \in C$ such that $-1 = x + ay$. If $y = 0$, then $x = -1 \in C$, a contradiction. If $y \neq$ then rearranging $-1 = x + ay$ gives

$$-a = \left(\frac{1}{y}\right)^2 \cdot y \cdot (1 + x).$$

Then $\left(\frac{1}{y}\right)^2$ since it is a square, $x, y \in C$ by assumption, and $1 \in C$ since it is also a square. Thus the product $\left(\frac{1}{y}\right)^2 \cdot y \cdot (1+x)$ is in C, contradicting that $-a \notin C$. The second property simply follows from $C \subseteq C[a]$. Finally, $C[a]$ is clearly closed under addition, and it is closed under multiplication since

$$(x + ay)(x' + ay') = (xx' + a^2yy') + a(xy' + x'y)$$

where both $(xx' + a^2yy') \in C$ and $(xy' + x'y) \in C$. □

Proof of Theorem. Since the union of an increasing chain of proper cones is a proper cone, Zorn's Lemma implies that there exists a maximal proper cone $\overline{C} \subset K$ that contains C. $\overline{C}$ satisfies the condition that for all $a \in K$ either $a \in \overline{C}$ or $-a \in \overline{C}$, otherwise $\overline{C} \subsetneq \overline{C}[a]$ which contradicts the maximality of of $\overline{C}$. Taking $P := \overline{C} - \{0\}$, P clearly satisfies the three conditions of Definition 1.1. □

Corollary 1.12. *Let K be a field. The following are equivalent:*

(i) -1 *is not a sum of squares in* K

(ii) K *has a proper cone*

(iii) K *can be ordered*

(vi) for all $x_1, \ldots, x_n \in K$, $\sum_{i=1}^n x_i^2 = 0 \ \Rightarrow \ x_2 = \cdots = x_n = 0$

Proof. $(i) \Rightarrow (ii)$ Let $C := \{\sum_{i=1}^n x_i^2 \ : \ n \in \mathbb{N}, x_i \in K\}$. It is a proper cone.
$(ii) \Rightarrow (iii)$ This was proved in Theorem 1.10.
$(iii) \Rightarrow (iv)$ This was proved in Corollary 1.8.
$(iv) \Rightarrow (i)$ Also proved in Corollary 1.8. □

The next example is a formally real field that is not $\mathbb{R}$.

Example 1.13. Let K be an ordered field, ε be a variable, and let $K(\varepsilon) = \{p/q \ : \ p, q \in K[\varepsilon]\}$ be the fraction field of the polynomial ring in ε over K. We define an ordering on $K(\varepsilon)$ as follows: For a polynomial $p = \sum_{i=m}^{n} a_m \varepsilon^i \in K[\varepsilon]$ with $a_m \neq 0$ we have $p > 0$ iff $a_m > 0$. For $p/q \in K(\varepsilon)$ we have $p/q > 0$ iff $pq > 0$. One can see that this defines a total ordering on $K(\varepsilon)$ compatible with the field operations.
We claim that for all $c \in K$, $c > 0$, we gave $\varepsilon < c$. Otherwise, if for some $c \in K$ we have $c < \varepsilon$, then $\varepsilon - c > 0$, which implies that $-c > 0$, i.e. $c < 0$.

The above example motivates the following definition:

Definition 1.14. Let (F, P) be an ordered field, and $K \subset F$ a subfiled. We say that $\varepsilon \in F$ *infinitesimal* over K, if $\varepsilon \in P$ and for all $c \in K \cap P$ we have $c - \varepsilon \in P$. An element $\delta \in F$ is *unbounded* over F if if $\delta \in P$ and for all $c \in K \cap P$ we have $\delta - c \in P$.

1.2 Real Closed Fields

Definition 1.15. The odered field (K, P) is *real closed* if

1. for all $a \in P$, a is a square, i.e. there exists $b \in K$ such that $a = b^2$;
2. for all $f \in K[x]$ with odd degree, f has a root in K.

The main theorem of this section is as follows:

Theorem 1.16. *Let (K, P) be an ordered field. Then the following are equivalent:*

(i) *K is real closed;*

(ii) *the factor ring $K[i] := K[x]/\langle x^2 + 1\rangle$ is an algebraically closed field;*

(iii) *K has the* Intermediate Value Property*, i.e. for $p \in K[x]$ and $a < b \in K$, if $p(a)p(b) < 0$ then there exists $c \in (a, b)$ such that $p(c) = 0$;*

(iv) *K is formally real and no proper field extension over K is formally real.*

Partial Proof. $(i) \Rightarrow (ii)$ We don't prove this.

$(ii) \Rightarrow (i)$ Suppose $K[i] := K[x]/\langle x^2 + 1\rangle$ is an algebraically closed field. First we prove that every positive element of K is a square. Let $a \in P$ and consider $x^2 - a \in K[x] \subset K[i]$. Then there exists $c + di \in K[i]$ such that $(c + di)^2 - a = 0$. But $(c + di)^2 = (c^2 - d^2) + 2cdi$, so $cd = 0$. If $c = 0$ then $a = -d^2 < 0$, a contradiction. Thus $d = 0$, so $a = c^2$, and a is a square in K.
Next we prove that any odd degree polynomial have a root in K. Let $f \in K[x]$ with odd degree. Then f factors into linear factors over $K[i]$. Suppose $x-(c+di)$ divides f over $K[i]$ for some $c, d, \in K$, and assume that $d \neq 0$. The polynomial $(x - c)^2 + d^2$ has no roots in K, otherwise $c + di \in K$ or $c - di \in K$, which would imply that $i \in K$ as $d \neq 0$, contradicting that $K[x]/\langle x^2 + 1\rangle$ is a field. Thus $(x - c)^2 + d^2$ is irreducible over K and divisible by $x - (c + di)$, so $(x - c)^2 + d^2$ must also divide f. This shows that the roots of f in $K[i] \setminus K$ come in pairs with their "conjugates", so an odd degree polynomial must have at least one root in K.

$(ii) \Rightarrow (iii)$ If $K[i] := K[x]/\langle x^2 + 1\rangle$ is an algebraically closed field, then all $f \in K[x]$ factors into a product of irreducible quadratic and linear polynomials over K. If q is a monic irreducible quadratic polynomial, then $q = (x-c)^2 + d^2$ for some $c, d \in K$, with $d \neq 0$, so q is positive for all $x \in K$. Thus there must be a linear divisor $l = x - c$ of f such that $l(a)l(b) < 0$, which implies that $c \in K$ is a root of f and $c \in (a, b)$.

(ii) $\Rightarrow$ *(iv)* If $K[i] := K[x]/\langle x^2+1\rangle$ is an algebraically closed field, then $x^2 + 1$ is an irreducible polynomial over K, so -1 is not a square in K. Also, -1 is not a sum of squares, since every sum of squares is a square, for example for arbitrary $a, b \in K$, $a^2 + b^2 = (c^2 + d^2)^2$ where $c + di$ is a solution of $x^2 - (a + bi)$ in $K[i]$. To prove that K has no proper field extension that is formally real, we show that $K[i]$ is the only proper filed extension of K, and that is not formally real. This is true, since the only non-linear irreducible polynomials are quadratic of the form $q = (x - c)^2 + d^2$ for $d \neq 0$, and $K[x]/\langle q\rangle = K[c + di] = K[i]$.

(iii) $\Rightarrow$ *(i)* To prove this, we will use Lemma 1.17 below. To prove that $a \in P$ is a square, consider $p = x^2 - a$. Then $p(0) < 0$. On the other hand, by Lemma 1.17, taking $\xi := 2a + 2$ we get that $\xi > 0$ and thus $p(\xi) > 0$. Thus by the intermediate value theorem there exists $b \in (0, \xi)$ such that $b^2 = a$. Similarly, to prove that an odd degree polynomial $p = a_n x^2 + \cdots + a_0$ with $a_n \neq 0$ and n odd, has a root in K, take $\xi_1 := 2\sum_{i=0}^{n} \left|\frac{a_i}{a_n}\right| + 1$ and $\xi_2 := -\xi_1$. Then again by Lemma 1.17, $p(\xi_1) > 0$ and $p(\xi_2) < 0$, so there is $c \in (\xi_2, \xi_1)$ such that $p(c) = 0$.

(vi) $\Rightarrow$ *(i)* (outline) Suppose (K, P) is an ordered field, but no proper field extension of K is formally real. Suppose there is $a \in P$ which is not a square in K. Then $x^2 - a$ is an irreducible polynomials, so $K[\sqrt{a}] := K[x]/\langle x^2 - a\rangle$ is a proper field extension of K. We can define an ordering on $K[\sqrt{a}] = \{c + d\sqrt{a} \ : \ c, d, \in K\}$ by $c + d\sqrt{a} > 0$ when either $c, d \in P$, or if $c \in P$ $d \notin P$ then $c^2 - ad^2 \in P$, or if $d \in P$ $c \notin P$ then $-c^2 + ad^2 \in P$. Let P' be the positive elements of $K[\sqrt{a}]$ defined this way. Clearly, $0 \notin P'$, P' has the trichotomy property. The assumption that $a \in P$ also implies that P' is closed under addition and multiplication, but we are not going to prove that here. Similarly, one can prove that if there exists a non-linear irreducible polynomial p of odd degree over K, then $K[x]/\langle p\rangle$ is a proper field extension that can be ordered. □

The following lemma was used in the proof above:

Lemma 1.17. *Let (K, P) be an ordered field and $p(x) = a_n x_n + a_{n-1}x^{n-1} + \cdots + a_0 \in K[x]$ with $a_n \neq 0$. Let $\xi \in K$ such that*

$$|\xi| > \max(2, 2\sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right|) \geq 2\sum_{i=0}^{n} \left|\frac{a_i}{a_n}\right|.$$

Then $\operatorname{sign}(p(\xi)) = \operatorname{sign}(a_n \xi^n)$.

Proof. Assume $|\xi| > \max(2, 2\sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right|)$. Then

$$\begin{aligned}
\frac{p(\xi)}{a_n \xi^n} &= 1 + \sum_{i=0}^{n-1} \frac{a_i}{a_n}\, \xi^{i-n} \\
&\geq 1 - \sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right| |\xi|^{i-n} \\
&\geq 1 - \left(\sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right|\right) \left(\sum_{i=0}^{n-1} |\xi|^{i-n}\right) \\
&\geq 1 - \frac{1}{2}\,|\xi| \left(\sum_{i=0}^{n-1} |\xi|^{i-n}\right) \\
&= 1 - \frac{1}{2}\left(1 + |\xi|^{-1} + \cdots + |\xi|^{n+1}\right) > 0,
\end{aligned}$$

where the last inequality follows for $|\xi| > 2$. □

Corollary 1.18. *If K is real closed then*

(a) the monic irreducible polynomials over K are linear and quadratic, and in the letter case they have the form $(x-c)^2 + d^2 = x^2 + ax + b$ for $a, b, c, d \in K$ such as $d \neq 0$ and $a^2 - 4b > 0$;

(b) K has a unique ordering.

(c) for $p \in K[x]$ and $a < b \in K$, if $p(x) \neq 0$ for $x \in (a, b)$ then $\operatorname{sign}(p(x))$ is constant on (a, b).

Proof. (a) we proved above that all irreducible polynomials are linear and quadratic, and it is an exercise to prove that the quadratic ones have the claimed form.
(b) Define $P = \{a \in K \ : \ \exists b \; b^2 = a\}$. Then if $P' \subset K$ is another ordering of K, then we proved above that $P \subseteq P'$. On the other hand, for $a \in P'$ either $a \in P$ or $-a \in P$. If $-a \in P$ then $-a$ is a square, which would imply that $-a \in P'$, a contradiction. Thus $a \in P$, proving that $P' \subset P$, so they must be equal.
(c) Follows from Intermediate Value Property. □

Remark 1.19. Note that for example $\mathbb{Q}[\sqrt{2}]$ has two possible orderings, one where $\sqrt{2}$ is positive and one where it is negative.

Definition 1.20. Let K be real closed, $a \in K$, $p \in K[x]$. The the *sign of p to the right of a* $\operatorname{sign}_r(p(a)) := \operatorname{sign}(p(x))$ for $x \in (a, b)$ such that p does not vanish on (a, b).
Similarly we can define $\operatorname{sign}_l(p(a))$, the sign to the left of a.
$\operatorname{sign}(p(\infty)) := \operatorname{sign}(p(M))$ such that M is sufficiently large so that p does not vanish on (M, ∞).

Proposition 1.21. *Let K be real closed. Suppose $r \in K$ is a root of $p \in K[x]$ of multiplicity μ, i.e. $p = (x-r)^\mu \cdot q$ and $q(r) \neq 0$. Then*

$$\operatorname{sign}_r(p(r)) = \operatorname{sign}(p^{(\mu)}(r)), \quad \operatorname{sign}_l(p(r)) = (-1)^\mu \operatorname{sign}(p^{(\mu)}(r)).$$

Proof.

$$\operatorname{sign}_{r,l}((p(r)) = \operatorname{sign}_{r,l}\left((x-r)^\mu|_{x=r}\right) \cdot \operatorname{sign}(q(r)).$$

Then the claim follows from $q(r) = p^{(\mu)}(r)$ and

$$\operatorname{sign}_r\left((x-r)^\mu|_{x=r}\right) = +1, \quad \operatorname{sign}_l\left((x-r)^\mu|_{x=r}\right) = (-1)^\mu.$$

□

Theorem 1.22 (Rolle Theorem). *Let K be real closed, $p \in K[x]$, $a < b \in K$. If $p(a) = p(b) = 0$ then there exists $c \in (a, b)$ such that $p'(c) = 0$.*

Proof. We can assume that p does not vanish on (a, b). Assume that $p = (x-a)^\mu (x-b)^\nu q$ and q does not vanish on $[a, b]$. Then

$$p' = (x-a)^{\mu-1}(x-b)^{\nu-1} q_1,$$

where

$$q_1 = m(x-b)q + n(x-a)q + (x-a)(x-b)q'.$$

Therefore, $q_1(a) = m(a-b)q(a)$ and $q_1(b) = m(b-a)q(b)$, so q_1 has opposite signs in a and b. By the Intermediate Value Property there exists $c \in (a, b)$ such that $q_1(c) = 0$, so $p'(c) = 0$ as well. □

Corollary 1.23 (Mean Value Theorem). *Let K be real closed, $p \in K[x]$, $a < b \in K$. Then there exists $c \in (a, b)$ such that*

$$p(b) - p(a) = (b-a)\, p'(c).$$

Proof. Apply Rolle Theorem to

$$q(x) := (p(b) - p(a))(x-a) - (b-a)(p(x) - p(a)).$$

□

Corollary 1.24. *Let K be real closed, $p \in K[x]$, $a < b \in K$. If p' is positive on (a, b) then p is increasing over $[a, b]$.*

2 Isolating Real Roots of Univariate Polynomials

Polynomial equations are used throughout mathematics. When solving polynomials many questions arise such as: Are there any real roots? If so, how many? Where are they located? Are these roots positive or negative? Depending on the problem being solved sometimes a rough estimate for the interval where a root is located is enough.

There are many methods that can be used to answer these questions. We will focus on Descartes' Rule of Signs, the Budan-Fourier theorem and Sturm's theorem. Descartes' Rule of Signs traditionally is used to determine the possible number of positive real roots of a polynomial. This method can be modified to also find the possible negative roots for a polynomial. The Budan-Fourier theorem takes advantage of the derivatives of a polynomial to determine the number of possible number of roots. While Sturm's theorem uses a blend of derivatives and the Euclidean Algorithm to determine the exact number of roots.

In some cases, an interval where a root of the polynomial exists is not enough. Two methods, Horner and Newton's methods, to numerically approximate roots up to a given precision are also discussed. We will also give a real world application that uses Sturm's theorem to solve a problem.

In this section $\mathbb{R}$ denotes the field of real numbers, but all claims are true for any real closed field.

2.1 Descartes's Rule of Signs

There are multiple ways to determine the number of roots of a polynomial. Descartes's Rule of Signs helps determine the number of positive or negative roots of a polynomial with real coefficients.

Theorem 2.1. *(Descartes's Rule of Signs) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients, where $a_i \in \mathbb{R}$, $i = 0, \ldots, n$, and a_n and a_0 are nonzero. Let v be the number of changes of signs in the sequence $(a_0, \ldots, a_n)$ of its coefficients and let p be the number of its real positive roots, counted with their orders of multiplicity, then there exists some nonnegative integer m such that*

$$p = v - 2m$$

(c.f. [Mig92]).

Proof: Let the number of positive roots of the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

be denoted by p and let v denote the number of sign variations of the coefficients. Since a_n is non zero, we can divide all the above coefficients by this number. This will not change the number of sign variations nor will it change the number of positive roots but it will allow us to assume that the leading coefficient of our polynomial is 1. We therefore can assume that $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where for some (possibly different) a_i. Let $r_1, \ldots, r_p$ be the positive roots (each listed as many times as its multiplicity). We then have

$$f(x) = g(x) \prod_{i=1}^{p} (x - r_i)$$

where $g(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ is a polynomial with no positive roots.

We will first show that p and v are both even or both odd. To see this first note that b_0 must be positive. If b_0 where negative, then $g(0) = b_0$ would be negative while for x sufficiently large $g(x)$ is positive. By the Intermediate Value Theorem, $g(x)$ would then have a positive root, a contradiction. Therefore b_0 is positive. We have that

$$a_0 = b_0 \left((-1)^p \prod_{i=1}^{p} r_i \right)$$

so a_0 is positive if p is even and a_0 is negative if p is odd. Since the leading coefficient of f is 1, and so is positive, the number of sign changes must be even when a_0 is positive and odd when a_0 is negative. This allows us to conclude that p and v are both even or both odd.

We now will show that $p \leq v$. We will do this by induction on n, the degree of f. If $n = 1$ we have that $f(x) = x + a_0$ (recall we are assuming that the leading coefficient is 1). There is only one root of this polynomial, $r_1 = -a_0$. If $a_0 > 0$, then f has no sign changes, $v = 0$, and its only root is negative, $p = 0$, so $v = p$. If $a_0 < 0$, then f has one sign change, $v = 1$, and on positive root, $p = 1$, so $v = p$ again.

Now assume that $n > 1$. Let q denote the number of positive roots of $f'(x)$ and w denote the number of sign changes of the coefficients of f'. Since

$$f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-1} + \ldots + a_1$$

the number of sign changes is at most v, so $w \leq v$. It is possible that $a_1 = 0$ and so our induction hypothesis does not directly apply. If this is the case, we can divide $f'(x)$ by some power x^k of x and yield a new polynomial $h(x) = f'(x)/x^k$ whose constant term is nonzero. Note that the number of sign changes in $h(x)$ is still w and the number of positive roots is still q. Applying the induction hypothesis we have $q \leq w$. Rolle's Theorem implies that $p - 1 \leq q$. Therefore

$$p - 1 \leq q \leq w \leq v.$$

We therefore have that $p - 1 \leq v$. We cannot have that $p - 1 = v$ since p and q are both even or both odd. Therefore $p \leq v$, the desired conclusion (c.f. [Con57]). □

It is assumed that the polynomials are written in descending powers of x. We will represent the coefficients of the polynomial $f(x)$ in the sequence $\{a_n, a_{n-1}, \ldots, a_1, a_0\}$. Some of the coefficients of the real polynomial may be zero. *We will only be considering the nonzero coefficients.* Let any coefficient $a_i < 0$ be represented by -1 and $a_i > 0$ be represented by 1. In order to gain a better understanding of Descartes's rule, we'll begin with an example.

Example 2.2. Consider the polynomial

$$f(x) = x^3 - 3x^2 - 4x + 13.$$

For this polynomial, the signs for the leading coefficients are represented by $(1, -1, -1, 1)$ resulting in two sign changes, therefore $v = 2$. According to Descartes's Rule of Signs, $f(x)$ has either

$$\begin{aligned} r &= 2 - 2(0) = 2 \text{ or} \\ r &= 2 - 2(1) = 0 \end{aligned}$$

positive roots. Hence, there is a maximum of 2 positive real roots and a minimum of 0 positive real root.

Example 2.3. Now, consider the polynomial

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5.$$

The signs for the leading coefficients are represented by $(1, -1, 1, 1, -1, 1)$, resulting in four sign changes, $v = 4$. According to Descartes's Rule of Signs, the polynomial has either 4, 2, or no positive roots. So, there is a maximum of 4 positive roots.

Example 2.4. Consider the polynomial

$$f(x) = x^6 - x^4 + 2x^2 - 3x - 1.$$

For this polynomial, the signs for the leading coefficients are represented by $(1, -1, 1, -1, -1)$ resulting in three sign changes, therefore $v = 3$. According to Descartes's Rule of Signs, $f(x)$ has either

$$\begin{aligned} r &= 3 - 2(0) = 3 \text{ or} \\ r &= 3 - 2(1) = 1 \end{aligned}$$

positive roots. Hence, there is a maximum of 3 positive real roots and a minimum of 1 positive real root.

It was mentioned that Descartes's Rule of Signs can be used to determine the positive and the negative roots of a polynomials with real coefficients.

Exercise 2.5. How can the Descartes's Rule of Signs be used to determine the number of negative roots for a polynomial with real coefficients? Hint: Use $f(-x)$.

So far all of our examples have had at least two sign changes. Now, let us think about what happens when there are only one or no sign changes in the polynomial (i.e. $v = 1$ or $v = 0$). Recall, that we can only have a nonnegative number of roots, $r \geq 0$, and that r can only be reduced by even values.

Exercise 2.6. How many roots polynomials have with only one sign change? no sign change?

Exercise 2.7. Determine the number of possible positive and negative roots for the following polynomials using Descartes's Rule of Signs.

1. $f(x) = x^4 - 2x^3 + 4x^2 - 3x + 1$.
2. $q(x) = x^9 + 3x^8 - 5x^3 + 4x + 6$.
3. $p(x) = 2x^3 + 5x^2 + x + 1$.
4. $f(x) = 3x^4 + 10x^2 + 5x - 4$.

In the statement of Descartes's Rule of Signs, it is assumed that $a_0 \neq 0$. What happens if $a_0 = 0$? Let us assume that $a_0 = 0$, then

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$$

where $a_i \in \mathbb{R}$, $i = 1, \ldots, n$. Can we still use Descartes's Rule of Signs? Yes!

Exercise 2.8. Given that $a_0 = 0$, show how Descartes's Rule of Signs can still be used to determine the positive (and negative) real roots for the polynomial. Hint: $x = 0$ is not considered a positive nor negative root. Divide.

2.2 Budan-Fourier Theorem

There are times when we want to know more than the number of real roots of a polynomial. We would prefer to know the exact values of the roots or an approximation. In order to determine real roots, we must know where they are located on the real line. A theorem used to determine an interval which contains a root is the Budan-Fourier theorem.

Theorem 2.9. *(Budan-Fourier Theorem) Let*

$$f(x) = 0$$

be a nonconstant polynomial of degree n with real coefficients. Designate by $v(c)$ the number of changes of signs in the sequence

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$$

when $x = c$, where c is any real number. Then the number of zeros of f in the interval $(a, b]$, counted with their orders of multiplicity is equal to

$$v(a) - v(b) - 2m, \quad \textit{for some } m \in \mathbb{N}$$

(c.f. [Con43]).

Proof: Let the interval $(a, b]$ be denoted by I. Let p equal the number of roots of $f(x) = 0$ on the interval I.

It can be seen that p is equal to the number of positive non-zero roots of

$$f(x+a) = f(a) + f'(a)x + \frac{1}{2}f''(a)x^2 + \cdots + \frac{1}{n!}f^{(n)}(a)x^n = 0 \tag{1}$$

minus the number of positive non-zero roots of

$$f(x+b) = f(b) + f'(b)x + \frac{1}{2}f''(b)x^2 + \cdots + \frac{1}{n!}f^{(n)}(b)x^n = 0. \tag{2}$$

This follows from the fact that equations (1) and (2) are obtained by diminishing the roots of $f(x) = 0$ by a and b respectively.

By Descartes' rule of signs, it follows that (1) has $v(a) - 2h_1$ positive roots and that (2) has $v(b) - 2h_2$ positive roots, for integers $h_1 \geq 0$ and $h_2 \geq 0$. Therefore

$$p = (v(a) - 2h_1) - (v(b) - 2h_2) = v(a) - v(b) - 2m, \tag{3}$$

where $m = h_1 - h_2$. We only need to prove that $m \geq 0$.

The proof that $m \geq 0$ will be effected by mathematical induction. It can be verified that $m \geq 0$ if $f(x)$ has degree one. Hence it will be sufficient to show that this inequality holds for every equation of degree n if the corresponding inequality holds for every equation of degree $n - 1$. We shall therefore base the argument to follow upon the assumption that the specified inequality is valid for every equation of degree $n - 1$.

Now let q denote the number of roots of the equation $f'(x) = 0$ in the interval I, and let $v'(c)$ denote the number of variations of sign in the sequence

$$f'(x), f''(x), \ldots, f^{(n)}(x),$$

when $x = c$, where c is any real number. By a similar argument to that which led to (3), it can be shown that

$$q = v'(a) - v'(b) - 2m'$$

where m' is a positive integer or zero. It can also be determined that $m' \geq 0$ by assumption.

It follows from Rolle's theorem that $q \geq p - 1$, say $q = p - 1 + s$ where $s \geq 0$ is an integer. We also note that

$$v(a) = v'(a) \quad \text{or} \quad v(a) = v'(a) + 1,$$

and

$$v(b) = v'(b) \quad \text{or} \quad v(b) = v'(b) + 1.$$

The possible combinations are

$$\begin{aligned}
v(a) - v(b) &= v'(a) - v'(b) \\
v(a) - v(b) &= v'(a) - (v'(b) + 1) = v'(a) - v'(b) - 1 \\
v(a) - v(b) &= (v'(a) + 1) - v'(b) = v'(a) - v'(b) + 1 \\
v(a) - v(b) &= (v'(a) + 1) - (v'(b) + 1) = v'(a) - v'(b).
\end{aligned}$$

Hence,

$$v(a) - v(b) \geq v'(a) - v'(b) - 1. \tag{4}$$

Consider the first case in which $v(a) - v(b) > v'(a) - v'(b) - 1$. Then

$$v(a) - v(b) \geq v'(a) - v'(b) = q + 2m' = p - 1 + s + 2m'. \tag{5}$$

Now if $s + 2m' \neq 0$, we have $v(a) - v(b) \geq p$. It follows from (3) that $m \geq 0$, as was to be proved.

If $s + 2m' = 0$, we have $v(a) - v(b) \geq p - 1$. But it follows from (3) that $v(a) - v(b) - p$ is even, and hence $v(a) - v(b) \neq p - 1$. Consequencely $v(a) - v(b) > p - 1$, and therefore $v(a) - v(b) \geq p$. Again it follows from (3) that $m \geq 0$.

Now consider the case in which the two members of (4) are equal. This is possible only if $v(a) = v'(a)$, and

$$v(b) = v'(b) + 1. \tag{6}$$

It will be shown later that, if (6) holds, then $q \geq p$. Assuming this fact for the moment, and letting $q = p + t$, where $t \geq 0$ is an integer, we have

$$v(a) - v(b) = v'(a) - v'(b) - 1 = q + 2m' - 1 = p + t + 2m' - 1. \tag{7}$$

The first and last members of (7) are essentially the same in (5). And by an argument similar to that developed in connection with (5) it can be shown that $m \geq 0$ in the case under consideration.

To complete the proof we need to show that $q \geq p$ if (6) holds. In this case $f(b) \neq 0$, and if $f'(b) \neq 0$, then $f(b)$ and $f'(b)$ must have different signs. Now let r be the greatest root of $f(x) = 0$ in the interval I. Then $f'(x)$ must vanish at least once in the interval $r < x \leq b$, or otherwise $f(b)$ and $f'(b)$ would be the same sign. Hence, the equation $f'(x) = 0$ has in the interval I the $p-1$ roots vouched for by Rolle's theorem, and at least one additional root in the interval $r < x \leq b$. Therefore $q \geq p$ (c.f. [Con43]). $\square$

We will expand on previous examples to demonstrate the Budan-Fourier theorem.

Example 2.10. Consider the polynomial in Example 2.2

$$\begin{aligned}
f(x) &= x^3 - 3x^2 - 4x + 13, \\
f'(x) &= 3x^2 - 6x - 4, \\
f''(x) &= 6x - 6, \\
f^{(3)}(x) &= 6.
\end{aligned}$$

Hence, we get Table 1.

In Table 1, the rows $x = -\infty$ and $x = \infty$ can be interpreted as follows. As x approaches positive infinity, any nonzero polynomial is eventually always positive or always negative. We denote $v(\infty)$ this sign. The variation $v(-\infty)$ is defined in a similar manner.

By Descartes's Rule of Signs there are either 2 or no positive real roots and one negative real root. From Table 1 we get,

$$\begin{aligned}
v(-3) - v(-2) - 2m &= 3 - 2 - 2m = 1, \\
v(-2) - v(-1) - 2m &= 2 - 2 - 2m = 0, \\
v(-1) - v(0) - 2m &= 2 - 2 - 2m = 0, \\
v(0) - v(1) - 2m &= 2 - 2 - 2m = 0, \\
v(1) - v(2) - 2m &= 2 - 2 - 2m = 0, \\
v(2) - v(3) - 2m &= 2 - 0 - 2m = 2 - 2m.
\end{aligned}$$

Table 1: Budan-Fourier Sign Variations

x	$f(x)$	$f'(x)$	$f''(x)$	$f^{(3)}(x)$	Variations
$-\infty$	$-$	$+$	$-$	$+$	3
-3	$-$	$+$	$-$	$+$	3
-2	$+$	$+$	$-$	$+$	2
-1	$+$	$+$	$-$	$+$	2
0	$+$	$-$	$-$	$+$	2
1	$+$	$-$	$+$	$+$	2
2	$+$	$-$	$+$	$+$	2
3	$+$	$+$	$+$	$+$	0
∞	$+$	$+$	$+$	$+$	0

Using the Budan-Fourier theorem, we can determine where these real roots are located. The Budan-Fourier theorem shows that there are no real roots on the interval $(-\infty, -3]$ since $v(a) - v(b) - 2m = 0$ for all $a, b \in (-\infty, -3]$. Since $f^{(i)}(3) \geq 0$ for $i = 0, \ldots, 3$, the polynomial has no real roots in the interval $[3, \infty)$. By the Budan-Fourier theorem, any of the possible 2 or no positive real roots are on the interval $[2, 3]$. There is also a negative real root on the interval $[-3, -2]$. Again note that the exact number and location of the real roots will take some additional investigation: one can compute that the polynomial has 3 real roots, they are approximately -2.0989, 2.3569, 2.6920.

Example 2.11. Consider the polynomial in Example 2.3

$$\begin{aligned}
f(x) &= x^5 - x^4 + 3x^3 + 9x^2 - x + 5, \\
f'(x) &= 5x^4 - 4x^3 + 9x^2 + 18x - 1, \\
f''(x) &= 20x^3 - 12x^2 + 18x + 18, \\
f^{(3)}(x) &= 60x^2 - 24x + 18, \\
f^{(4)}(x) &= 120x - 24, \\
f^{(5)}(x) &= 120.
\end{aligned}$$

Hence, we get Table 2.

Table 2: Budan-Fourier Sign Variations

x	$f(x)$	$f'(x)$	$f''(x)$	$f^{(3)}(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$	Variations
$-\infty$	$-$	$+$	$-$	$+$	$-$	$+$	5
-3	$-$	$+$	$-$	$+$	$-$	$+$	5
-2	$-$	$+$	$-$	$+$	$-$	$+$	5
-1	$+$	$-$	$-$	$+$	$-$	$+$	4
0	$+$	$-$	$+$	$+$	$-$	$+$	4
1	$+$	$+$	$+$	$+$	$+$	$+$	0
∞	$+$	$+$	$+$	$+$	$+$	$+$	0

By Descartes's Rule of Signs there are either 4, 2, or no positive real roots and one negative real root. From Table 2 we get,

$$\begin{aligned} v(-3) - v(-2) - 2m &= 5 - 5 - 2m = 0, \\ v(-2) - v(-1) - 2m &= 5 - 4 - 2m = 1, \\ v(-1) - v(0) - 2m &= 4 - 4 - 2m = 0, \\ v(0) - v(1) - 2m &= 4 - 0 - 2m = 4 - 2m. \end{aligned}$$

Since $f^{(i)}(1) \geq 0$ for $i = 0, 1, \ldots, 5$, the polynomial has no real roots in the interval $[1, \infty)$. By the Budan-Fourier theorem, any of the possible $4, 2$, or no positive real roots are on the interval $[0, 1]$. Since $v(a) - v(b) - 2m = 0$ for all $a, b \in (-\infty, -3]$, the Budan-Fourier theorem also shows that there are no real roots on the interval $(-\infty, -3]$. However, there is a negative real root on the interval $[-2, -1]$. Again note that the exact number and location of the real roots will take some additional investigation: one can compute that the polynomial has only 1 real roots, which is approximately -1.6073.

Example 2.12. Consider the polynomial in Example 1.1.3

$$\begin{aligned} f(x) &= x^6 - x^4 + 2x^2 - 3x - 1, \\ f'(x) &= 6x^5 - 4x^3 + 4x - 3, \\ f''(x) &= 30x^4 - 12x^2 + 4, \\ f^{(3)}(x) &= 120x^3 - 24x, \\ f^{(4)}(x) &= 360x^2 - 24, \\ f^{(5)}(x) &= 720x, \\ f^{(6)}(x) &= 720. \end{aligned}$$

Hence, we get Table 3.

Table 3: Budan-Fourier Sign Variations

x	$f(x)$	$f'(x)$	$f''(x)$	$f^{(3)}(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$	$f^{(6)}(x)$	Variations
$-\infty$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	6
-2	$+$	$-$	$+$	$-$	$+$	$-$	$+$	6
-1	$+$	$-$	$+$	$-$	$+$	$-$	$+$	6
0	$-$	$-$	$+$	$+$	$-$	$+$	$+$	3
1	$-$	$+$	$+$	$+$	$+$	$+$	$+$	1
2	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0
∞	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0

By Descartes's Rule of Signs there are either 3 or 1 positive real roots and either 3 or 1 negative roots. From Table 3 we get,

$$v(-2) - v(-1) - 2m = 6 - 6 - 2m = 0 \tag{8}$$

$$v(-1) - v(0) - 2m = 6 - 3 - 2m = 3 - 2m \tag{9}$$

$$v(0) - v(1) - 2m = 3 - 1 - 2m = 2 - 2m, \tag{10}$$

$$v(1) - v(2) - 2m = 1 - 0 - 2m = 1. \tag{11}$$

Using the Budan-Fourier theorem, we can determine where these real roots are located. The Budan-Fourier theorem shows that there are no real roots on the intervals $(-\infty, -1]$ since $v(a) - v(b) - 2m = 0$ for all $a, b \in (-\infty, -1]$. Since, $f^{(i)}(1) \geq 0$ for $i = 0, 1, \ldots 6$ the polynomial also has no real roots on the interval $[2, \infty)$. The Budan-Fourier theorem also shows that

- equation (9) has either 3 or 1 real roots on the interval $[-1, 0]$,
- equation (10) has either 2 or no real roots on the interval $[0, 1]$, and
- equation (11) has exactly one real root on the interval $[1, 2]$.

Note that in order to determine the exact number and location of the real roots, some additional investigations must be done: one can compute that the polynomial has 2 real roots, they are approximately -0.2821 and 1.2692.

Exercise 2.13. In Exercise 2.7 you determined the number of possible positive and negative roots for the following polynomials using Descartes's Rule of Signs. Now locate where these possible real roots are located on the number line.

1. $f(x) = x^4 - 2x^3 + 4x^2 - 3x + 1.$
2. $p(x) = 2x^3 + 5x^2 + x + 1.$
3. $f(x) = 3x^4 + 10x^2 + 5x - 4.$

Exercise 2.14. Show that Descartes's Rule of Signs is a special case of Budan-Fourier's Theorem.

Exercise 2.15. The behavior of the roots of nonzero polynomials at $x = \infty$ and $x = -\infty$ was mentioned earlier. Show that $v(\infty)$ is the sign of the coefficient of the highest degree term. Interpret $v(-\infty)$ in a similar way.

2.3 Sturm's Theorem

Before stating Sturm's theorem, we will first explain how to obtain the functions used by the theorem.

Let $f(x)$ be a polynomial with real coefficients and $f_1(x) := f'(x)$ its first derivative. We will modify the Extended Euclidean Algorithm by exhibiting each remainder as the negative of a polynomial. In order words, the remainder resulting from the division of $f(x)$ by $f_1(x)$ is $-f_2(x)$; the remainder resulting from the division of $f_1(x)$ by $f_2(x)$ is $-f_3(x)$; and so on. Continue the procedure until a constant remainder $-f_n$ is obtained. So, we have

$$\begin{aligned}
f(x) &= q_1(x)f_1(x) - f_2(x), \\
f_1(x) &= q_2(x)f_2(x) - f_3(x), \\
f_2(x) &= q_3(x)f_3(x) - f_4(x), \\
&\vdots \\
f_{n-2}(x) &= q_{n-1}(x)f_{n-1}(x) - f_n, \quad \text{where} \quad f_n = \text{ constant.}
\end{aligned}$$

Note that $f_n \neq 0$ if the polynomial $f(x)$ does not have a multiple roots. The sequence of polynomials

$$f(x),\ f_1(x),\ f_2(x),\ \cdots,\ f_{n-1}(x),\ f_n, \tag{12}$$

is called the *Sturm sequence*. The f_i are referred to as the *Sturm functions*.

If $x = c$ where $c \in \mathbb{R}$, then the number of variations of signs of sequence (12) is denoted by $v(c)$. Note that this $v(c)$ is different than the one used for the Budan-Fourier theorem. Be careful not to confuse the two notations. Any terms of the sequence that become 0 are dropped before counting the variations in signs.

In practice the computation needed to obtain Sturm functions is costly, especially if $f(x)$ is of high degree. However, once the functions are found, it is easy to locate the real roots of the equation $f(x) = 0$ by the Sturm's theorem (c.f. [Con57]).

Theorem 2.16. *(Sturm's Theorem) Let $f(x) = 0$ be an algebraic equation with real coefficients and without multiple roots. If a and b are real numbers, $a < b$, and neither a root of the given equation, then the number of real roots of $f(x) = 0$ between a and b is equal to $v(a) - v(b)$ (c.f. [Con57]).*

Proof: We will follow the proof given in *Algebra, Third Edition* by S. Lang, Addison-Wesley 1993. First note that the Sturm Sequence $\{f_0 = f, f_1 = f', \ldots, f_n\}$ satisfies the following four properties that follow from the definitions of the f_i:

1. The last polynomial f_n is a non-zero constant.
2. There is no point $x \in [a, b]$ such that $f_j(x) = f_{j+1}(x) = 0$ for any value $j = 0, \ldots, n-1$.
3. If $x \in [a, b]$ and $f_j(x) = 0$ for some $j = 1, \ldots, n-1$, then $f_{j-1}(x)$ and $f_{j+1}(x)$ have opposite signs.
4. We have $f_0(a) \neq 0$ and $f_0(b) \neq 0$.

Item 1. follows from the fact that f has no multiple roots. To verify Item 2. note that

$$f_j = q_i f_{j+1} - f_{j+2}. \tag{13}$$

Therefore, if $f_j(x) = f_{j+1}(x) = 0$, we would have $f_{j+2}(x) = 0$, and by induction, $f_n(x) = 0$, a contradiction. Item 3. follow from the fact that if $f_j(x) = 0$, then equation (13) implies that $f_{j-1}(x) = -f_{j+1}(x)$. Item 4. is part of the definition of a and b.

Let $r_1 < \ldots < r_s$ be a list of the zeros of the Sturm functions on the interval $[a, b]$, and we first assume that $a < r_1$ and $r_s < b$. Note that $v(x)$ is constant on any interval (r_i, r_{i+1}). Therefore it is enough to show that if $c < d$ are any numbers such that there is precisely one element r that is a root of some f_j with $c < r < d$, then $v(c) - v(d) = 1$ if r is a root of f and $v(c) - v(d) = 0$ if r is a not a root of f. The conclusion of the theorem will then follow from the fact that

$$v(a) - v(b) = [v(x_0) - v(x_1)] + [v(x_1) - v(x_2)] + \ldots + [v(x_{s-1}) - v(x_s)]$$

where the x_i are selected such that $r_i < x_i < r_{i+1}$ and $x_0 = a, x_s = b$.

Suppose that r is a root of some f_j. By Item 3. we have that $f_{j-1}(r)$ and $f_{j+1}(r)$ have opposite signs and by Item 2. these do not change when we replace r by c or d, thus $\text{sign}(f_i(c)) = \text{sign}(f_i(d))$ for all $i \neq j$, and $\text{sign}(f_j(c)) = -\text{sign}(f_j(d))$. Therefore, there will be one sign change in each of the triples

$$\{f_{j-1}(c), f_j(c), f_{j+1}(c)\} \qquad \{f_{j-1}(d), f_j(d), f_{j+1}(d)\}.$$

If r is not a roof of f, then we have that $v(c) = v(d)$. If r is a root of f then $f(c)$ and $f(d)$ have opposite signs. We also have that $f'(r) \neq 0$, so $f'(c)$ and $f'(d)$ have the same sign. If $f(c) < 0$ then f must be increasing so $f(d) > 0$, $f'(c) > 0$ and $f'(d) > 0$ and if $f(c) > 0$, then f must be decreasing so $f(d) < 0$, $f'(c) < 0$ and $f'(d) < 0$. Therefore there is one sign change in $\{f(c), f'(c)\}$ and none in $\{f(d), f'(d)\}$. So we have $v(c) = v(d) + 1$ (c.f. [Lan93]).

Finally, if $a = r_1$ or $r_s = b$, since $f(a) \neq 0$ and $f(b) \neq 0$, the above argument shows that $v(a) = v(x_0)$ for a sufficiently close $x_0 \leq a$ and similarly $v(b) = v(x_s)$ for some $x_s \geq b$. $\square$

Remark: Notice that Sturm's theorem is limited to polynomials that do not have multiple roots. In general, assume that $\gcd(f, f') = d$, a positive degree polynomial. In this case we can prove that if $a < b \in \mathbb{R}$ not roots of f then that $v(a) - v(b)$ is the number of roots in $[a, b]$ not counting multiplicities. To prove this, one can replace the Sturm sequence $(f_0 = f, f_1 = f', f_2, \ldots, f_n = d)$ by the sequence $(f_0/d, f_1/d, f_2/d, \ldots, f_n/d)$. Note that the sign variations of this new sequence at a and b are still $v(a)$ and $v(b)$ respectively, but now we can apply the proof of Sturm's Theorem for this sequence to prove that $v(a) - v(b)$ is equal to the number of roots of f/d in $[a, b]$, thus counting the roots without multiplicity.

In order to gain a better understanding of the theorem we will expand on previous examples.

Example 2.17. (C.f. [Con57], p.88) Consider the polynomial in Examples 2.2 and 2.10

$$f(x) = x^3 - 3x^2 - 4x + 13.$$

Use Sturm's theorem to isolate the roots of the equation $f(x) = 0$. We will be using **Maple** to calculate the remainders. The first derivative of $f(x)$ is

$$f_1(x) = 3x^2 - 6x - 4.$$

Using **Maple**, we can determine $f_2(x)$ which is the negative remainder of $f(x)$ divided by $f_1(x)$.

Maple Code:

```
> r:= rem(f,f_1,x,'q');
```

$$r := -\tfrac{14}{3}x + \tfrac{35}{3}$$

```
> f_2 := -r;
```

$$f_2 := \tfrac{14}{3}x - \tfrac{35}{3}$$

```
> q;
```

$$q := \tfrac{1}{3}x - \tfrac{1}{3}$$

Given that q is the quotient of f divided by f_1 and r is the remainder, we obtain

$$x^3 - 3x^2 - 4x + 13 = \left(\frac{1}{3}x - \frac{1}{3}\right)\left(3x^2 - 6x - 4\right) + \left(-\frac{14}{3}x + \frac{35}{3}\right).$$

Again using **Maple**, we can determine $f_3(x)$ which is the negative remainder of $f_1(x)$ divided by $f_2(x)$.

Maple Code:

```
>  r:= rem(f_1,f_2,x,'q');
```

$$r := -\frac{1}{4}$$

```
>  f_3 := -r;
```

$$f_3 := \frac{1}{4}$$

```
>  q;
```

$$q := \frac{9}{14}x + \frac{9}{28}$$

Given that q is the quotient of f_1 divided by f_2 and r is the remainder, we obtain

$$3x^2 - 6x - 4 = \left(\frac{9}{14}x + \frac{9}{28}\right)\left(\frac{14}{3}x - \frac{35}{3}\right) - \frac{1}{4}.$$

Therefore, the list of Sturm functions is as follows

$$\begin{aligned} f &= x^3 - 3x^2 - 4x + 13, \\ f_1 &= 3x^2 - 6x - 4, \\ f_2 &= \frac{14}{3}x - \frac{35}{3}, \\ f_3 &= \frac{1}{4}. \end{aligned}$$

The signs of Sturm functions for selected x are in Table 4. We can see from Table 4 that

Table 4: Sturm Function Sign Variations

x	f	f_1	f_2	f_3	Variations
$-\infty$	$-$	$+$	$-$	$+$	3
-3	$-$	$+$	$-$	$+$	3
-2	$+$	$+$	$-$	$+$	2
0	$+$	$-$	$-$	$+$	2
2	$+$	$-$	$-$	$+$	2
3	$+$	$+$	$+$	$+$	0
∞	$+$	$+$	$+$	$+$	0

$$\begin{aligned} v(-3) - v(-2) &= 3 - 2 = 1 \\ v(2) - v(3) &= 2 - 0 = 2 \end{aligned}$$

and the other combinations of $v(a) - v(b)$ are equal to zero. Therefore, by Sturm's Theorem we know there is a single root between -3 and -2 and that there are exactly two roots between 2 and 3.

Example 2.18. Consider the polynomial in Examples 2.3 and 2.11

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5.$$

Use Sturm's theorem to isolate the roots of the equation $f(x) = 0$. We will be using **Maple** to calculate the remainders. The first derivative of $f(x)$ is

$$f_1(x) = 5x^4 - 4x^3 + 9x^2 + 18x - 1.$$

Using **Maple** we will determine the remainder r and quotient q of f_i and f_{i+1} and assigning $-r$ to f_{i+2}. We will denote f_0 as f in all code. Dividing $f(x)$ by $f_1(x)$, we get

Maple Code:

```
> r:= rem(f,f_1,x,'q');
```

$$r := \tfrac{26}{25}x^3 + \tfrac{144}{25}x^2 - \tfrac{2}{25}x + \tfrac{124}{25}$$

```
> f_2 := -r;
```

$$f_2 := -\tfrac{26}{25}x^3 - \tfrac{144}{25}x^2 + \tfrac{2}{25}x - \tfrac{124}{25}$$

```
> q;
```

$$q := \tfrac{1}{5}x - \tfrac{1}{25}$$

Given that q is the quotient of f divided by f_1 and r is the remainder, we obtain

$$x^5 - x^4 + 3x^3 + 9x^2 - x + 5 = \left(\frac{1}{5}x - \frac{1}{25}\right) f_1(x) + \left(\frac{26}{25}x^3 + \frac{144}{25}x^2 - \frac{2}{25}x + \frac{124}{25}\right).$$

Now, divide $f_1(x)$ by $f_2(x)$ to get

Maple Code:

```
> r:= rem(f_1,f_2,x,'q');
```

$$r := \tfrac{25375}{169} + \tfrac{31250}{169}x^2 - \tfrac{1400}{169}x$$

```
> f_3 := -r;
```

$$f_3 := -\tfrac{25375}{169} - \tfrac{31250}{169}x^2 + \tfrac{1400}{169}x$$

```
> q;
```

$$q := -\tfrac{125}{26}x + \tfrac{5150}{169}$$

Given that q is the quotient of f_1 divided by f_2 and r is the remainder, we obtain

$$5x^4 - 4x^3 + 9x^2 + 18x - 1 = \left(-\frac{125}{26}x + \frac{5150}{169}\right) f_2(x) + \left(\frac{31250}{169}x^2 - \frac{1400}{169}x + \frac{25375}{169}\right).$$

Now, divide $f_2(x)$ by $f_3(x)$ to get

Maple Code:

```
> r:= rem(f_2,f_3,x,'q');
```

$$r := \frac{6487741}{9765625}x - \frac{478608}{1953125}$$

```
> f_4 := -r;
```

$$f_4 := -\frac{6487741}{9765625}x + \frac{478608}{1953125}$$

```
> q;
```

$$q := \frac{2197}{390625}x + \frac{7666516}{244140625}$$

Given that q is the quotient of f_2 divided by f_3 and r is the remainder, we obtain

$$-\frac{26}{25}x^3 - \frac{144}{25}x^2 + \frac{2}{25}x - \frac{124}{25} = \left(\frac{2197}{390625}x + \frac{7666516}{244140625}\right) f_3(x) + \left(\frac{6487741}{9765625}x - \frac{478608}{1953125}\right).$$

Lastly, divide $f_3(x)$ by $f_4(x)$ to get

Maple Code:

```
> r:= rem(f_3,f_4,x,'q');
```

$$r := -\frac{42900302734375}{249057889249}$$

```
> f_5 := -r;
```

$$f_5 := \frac{42900302734375}{249057889249}$$

```
> q;
```

$$q := \frac{305175781250}{1096428229}x + \frac{3796439453125000}{42090783283081}$$

Given that q is the quotient of f_3 divided by f_4 and r is the remainder, we obtain

$$-\frac{31250}{169}x^2 + \frac{1400}{169}x - \frac{25375}{169} = \left(-\frac{305175781250}{1096428229}x - \frac{3796439453125000}{42090783283081}\right) f_3(x) + \frac{42900302734375}{249057889249}.$$

Therefore, the list of Sturm functions is as follows

$$\begin{aligned}
f &= x^5 - x^4 + 3x^3 + 9x^2 - x + 5, \\
f_1 &= 5x^4 - 4x^3 + 9x^2 + 18x - 1, \\
f_2 &= -\frac{26}{25}x^3 - \frac{144}{25}x^2 + \frac{2}{25}x - \frac{124}{25}, \\
f_3 &= -\frac{31250}{169}x^2 + \frac{1400}{169}x - \frac{25375}{169}, \\
f_4 &= -\frac{6487741}{9765625}x + \frac{478608}{1953125}, \\
f_5 &= \frac{42900302734375}{249057889249}.
\end{aligned}$$

The signs of the Sturm functions for select significant values of x are in Table 5. We can see from Table 5 that

$$v(-2) - v(-1) = 3 - 2 = 1$$

and the other combinations of $v(a)-v(b)$ are equal to zero. Therefore, by Sturm's Theorem we know there is a single root between -2 and -1 and no others.

Table 5: Sturm Function Sign Variations

x	f	f_1	f_2	f_3	f_4	f_5	Variations
$-\infty$	$-$	$+$	$+$	$-$	$+$	$+$	3
-3	$-$	$+$	$-$	$-$	$+$	$+$	3
-2	$-$	$+$	$-$	$-$	$+$	$+$	3
-1	$+$	$-$	$-$	$-$	$+$	$+$	2
0	$+$	$-$	$-$	$-$	$+$	$+$	2
1	$+$	$+$	$-$	$-$	$-$	$+$	2
2	$+$	$+$	$-$	$-$	$-$	$+$	2
3	$+$	$+$	$-$	$-$	$-$	$+$	2
∞	$+$	$+$	$-$	$-$	$-$	$+$	2

Maple was used in all of the examples to compute Sturm functions. Depending on the degree of the polynomial we are given, this can be very tedious. There are calls in **Maple** that will generate the Sturm functions and another which will give the number of real roots of a polynomial on an interval.

- **sturm** - number of real roots of a polynomial in an interval
- **sturmseq** - Sturm sequence of a polynomial

Calling Sequence

```
> sturmseq(p,x);
> sturm(s,x,a,b);
```

Parameters

p	-	polynomial in x with rational or float coefficients
x	-	variable in polynomial p
a, b	-	rationals or floats such that $a \leq b$; a can be $-\infty$ and b can be ∞
s	-	Sturm sequence for polynomial p

Note: The interval excludes the lower endpoint a and includes the upper endpoint b (unless it is ∞). This is different from how the Sturm theorem is stated in Theorem 1.3.

Exercise 2.19. Use the call sequences in **Maple** mentioned above to generate Sturm functions for Examples 1.3.1, 1.3.2, and 1.3.3. How do the **Maple** generated Sturm functions differ from those in the examples? Does it change the intervals where the real roots can be found?

Exercise 2.20. Create a procedure in **Maple** that will output Sturm functions given a polynomial with real coefficients. Check your Sturm functions with those generated by the call sequence in **Maple**.

Exercise 2.21. In Exercise 1.2.1 you located the possible real roots on the number line using the Budan-Fourier theorem. Now, isolate the real roots of each of the following equations by means of Sturm's theorem:

1. $f(x) = x^4 - 2x^3 + 4x^2 - 3x + 1$.
2. $p(x) = 2x^3 + 5x^2 + x + 1$.
3. $f(x) = 3x^4 + 10x^2 + 5x - 4$.

2.4 Generalized Sturm's Theorem

The Generalized Sturm's Theorem can be used to find roots of polynomials that satisfy polynomial inequalities.

Theorem 2.22. *(Generalized Sturm's Theorem) Let $f(x), g(x) \in \mathbb{R}[x]$ and consider the Sturm sequence $[f_0 = f, f_1 = f'g, f_2, \ldots, f_n]$ defind by the Eucledian algorithm on f and $f'g$. As above, for $c \in \mathbb{R}$, $v(c)$ denotes the number of sing changes in the sequence $[f_0(c), f_1(c), \ldots, f_n(c)]$. If a and b are real numbers, $a < b$, and neither a of f, then*

$$v(a) - v(b) = \#\{c \in [a,b] \ : \ f(c) = 0 \text{ and } g(c) > 0\} - \#\{c \in [a,b] \ : \ f(c) = 0 \text{ and } g(c) < 0\}.$$

Proof: First note that similarly as in the proof of Sturm's theorem above we can divide up $[a,b]$ to $x_0 = a < x_1 < \cdots < x_s = b$ such that there is a unique root of f_j in each open interval (x_i, x_{i+1}). Also, the same argument as in the proof of Sturm's theorem we can see that if this root is not a root of f, then $v(x_i) = v(x_{i+1})$. Therefore, it is enough to prove that if there is a unique $c \in (a,b)$ such that$f(c) = 0$ then

$$v(a) - v(b) = \operatorname{sign}(g(c)).$$

Denote by

$$f(x) = (x-c)^r \varphi(x), \quad \text{and} \quad g(x) = (x-c)^s \psi(x)$$

where $r > 0$, $\varphi(c) \neq 0$ and $\psi(c) \neq 0$.

Case 1: First assume that $s = 0$, so $\psi(x) = g(x)$. We will consider the polynomial $f \cdot f' \cdot g$ because $v(a) - v(b) = \#\text{signchange}\,[f(a), f'g(a)] - \#\text{signchange}\,[f(b), f'g(b)]$, and

$$\#\text{signchange}\left[f(x), f'g(x)\right] = \begin{cases} 1 & \text{if } \operatorname{sign}(ff'g(x)) = -1 \\ 0 & \text{if } \operatorname{sign}(ff'g(x)) = 1 \end{cases}.$$

Since $f'(x) = r(x-c)^{r-1}\varphi(x) + (x-c)^r \varphi'(x)$ we get

$$f(x) \cdot f'(x) \cdot g(x) = (x-c)^{2r-2} \left(r(x-c)\varphi^2(x) g(x) + (x-c)^2 \varphi(x)\varphi'(x) g(x) \right).$$

Then $(x-c)^{2r-2} \geq 0$. Since

$$\lim_{x \to c} r\varphi^2(x) g(x) + (x-c)\varphi(x)\varphi'(x) g(x) = r\varphi^2(x) g(x),$$

therefore if x is sufficiently close to c we have that

$$\operatorname{sign}(ff'g(x)) = \operatorname{sign}\left(r(x-c)\varphi^2(x) g(x) + (x-c)^2 \varphi(x)\varphi'(x) g(x) \right) = \operatorname{sign}\left((x-c) g(c)\right).$$

This implies that

$$\operatorname{sign}(ff'g(a)) = -\operatorname{sign}(g(c)), \quad \operatorname{sign}(ff'g(b)) = \operatorname{sign}(g(c)).$$

Thus

$$\#\text{signchange}\left[f(a), f'g(a)\right] = \begin{cases} 1 & \text{if } g(c) > 0 \\ 0 & \text{if } g(c) < 0 \end{cases}, \quad \#\text{signchange}\left[f(b), f'g(b)\right] = \begin{cases} 1 & \text{if } g(c) < 0 \\ 0 & \text{if } g(c) > 0 \end{cases},$$

which implies that $v(a) - v(b) = \text{sign}(g(c))$, as claimed.

Case 2: $s > 0$. Then $(x-c)^r$ divides $\gcd(f, f'g)$, so we can replace the sequence $[f, f'g, f_2, \ldots, f_n]$ by $[f/(x-c)^r, f'g/(x-c)^r, f_2/(x-c)^r, \ldots, f_n/(x-c)^r]$. This new sequence will have the same number of variations of signs $v(a)$ and $v(b)$, but since $f/(x-c)^r$ has no zeroes in $[a, b]$, the sign variation $v(a) - v(b)$ must be $0 = g(c) = \text{sign}(g(c))$, as claimed. $\square$

3 Appendix

We will note a few facts from calculus that were used in the proofs above.

3.1 Rolle's Theorem

Theorem 3.1. *Let $f(x) = 0$ be an algebraic equation with real coefficients. Between two consecutive real roots a and b of this equation there is an odd number of roots of the equation $f'(x) = 0$. A root of multiplicity m is here counted as m roots (c.f. [Con57]).*

3.2 Intermediate Value Theorem

Theorem 3.2. *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$ (c.f. [Ste06], p.124).*

One of general uses of the Intermediate Value Theorem is to find real solutions to equations. A simple example from calculus will help clarify any questions.

Example 3.3. (C.f. [Ste06], p. 125) Show there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

on the interval $[1, 2]$.
Solution Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We want to show there exists a number c between 1 and 2 that is a solution to $f(x)$ such that $f(c) = 0$. Consider $a = 1$, $b = 2$ and $N = 0$ in the Intermediate Value Theorem. Then

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0.$$

Therefore, $f(1) < 0 < f(2)$, that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Since f is continuous function the Intermediate Value Theorem says there exists a number c between 1 and 2 such that $f(c) = 0$. So, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least root on the interval $(1, 2)$.

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