In Class Problems

MA 722 Spring 2012

1. **Euler identity and the kernel of the Jacobian matrix**
(a) Let F be a homogeneous polynomial of degree m in the variables $x_0, x_1, \ldots, x_n$. Prove the *Euler identity*:

$$\frac{\partial F}{\partial x_0}x_0 + \frac{\partial F}{\partial x_2}x_2 + \cdots + \frac{\partial F}{\partial x_n}x_n = mF(x_0, x_1, \ldots, x_n).$$

(b) Let $f = (f_1, \ldots, f_k)$ be a vector of k homogeneous polynomials of degrees $(d_1, \ldots, d_k)$ in the variables $\mathbf{x} = (x_0, x_1, \ldots, x_n)$. Let $Df(\mathbf{x})$ be the $k \times (n+1)$ matrix of partial differentials of f evaluated at $\mathbf{x}$. Assume that for some $\tilde{\mathbf{x}} = (\tilde{x}_0, \ldots, \tilde{x}_n) \in \mathbb{C}^{n+1}$ $f(\tilde{\mathbf{x}}) = 0$. Prove that the matrix-vector product

$$Df(\tilde{\mathbf{x}})\tilde{\mathbf{x}} = 0.$$

(c) Let f and $\tilde{\mathbf{x}}$ as in part (b). Prove that if $Df(\tilde{\mathbf{x}})\tilde{\mathbf{x}} = 0$ for some $\tilde{\mathbf{x}} \in \mathbb{C}^n$ then $f(\tilde{\mathbf{x}}) = 0$.

2. **Inverse Function Theorem implies Bezout's theorem.**
Let $(d) = (d_1, \ldots, d_n) \in \mathbb{N}^n_{>0}$ and $\mathcal{H}_{(d)} = \{f = (f_1, \ldots, f_n) \ : \ \forall i \ f_i$ homogeneous of degree $d_i\} \subset \mathbb{C}[x_0, \ldots, x_n]^n$. Consider the solution variety

$$V = \{(f, \mathbf{z}) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \ : \ f(\mathbf{z}) = 0\}.$$

Denote the projection $\pi : \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1}) \to \mathcal{H}_{(d)}, \quad \pi(f, \mathbf{z}) = f$, and its restriction to V by $\pi_V := \pi|_V$. Abusing the notation, for $\mathbf{z} = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ we denote by $\mathbf{z} = (z_0 : z_1 : \cdots : z_n) \in \mathbb{P}(\mathbb{C}^{n+1})$.

(a) Show that the derivative $D\pi(f, \mathbf{z})$ of π at $(f, \mathbf{z}) \in \mathcal{H}_{(d)} \times \mathbb{P}(\mathbb{C}^{n+1})$ is equal to the projection

$$\pi' : \mathcal{H}_{(d)} \times T_{\mathbf{z}} \to \mathcal{H}_{(d)}, \quad \pi'(h, \mathbf{w}) = h,$$

for $h \in \mathcal{H}_{(d)}$ and $\mathbf{w} \in T_{\mathbf{z}}$. Here $T_{\mathbf{z}} = \{\mathbf{y} \in \mathbb{C}^{n+1} \ : \ \mathbf{z} \cdot \overline{\mathbf{y}} = 0\}$ is the hyperplane of vectors orthogonal to the line going through $\mathbf{z}$ in $\mathbb{C}^{n+1}$, and note that $T_{\mathbf{z}}$ can be viewed as the tangent space of $\mathbb{P}(\mathbb{C}^{n+1})$ at $\mathbf{z}$.

(b) Show that the derivative $D\pi_V(f, \mathbf{z})$ of π_V at $(f, \mathbf{z}) \in V$ is the restriction of π' to the tangent space $T_{(f,\mathbf{z})}V = \{(h, \mathbf{w}) \in \mathcal{H}_{(d)} \times T_{\mathbf{z}} \mid h(\mathbf{z}) + Df(\mathbf{z})\mathbf{w} = 0\}$. Here $Df(\mathbf{z}) \in \mathbb{C}^{n \times (n+1)}$ is the Jacobian matrix of f at $\mathbf{z}$.

(c) Prove that if $\mathrm{rank}(Df(\mathbf{z})) = n$ then $D\pi_V(f, \mathbf{z})$ is invertible.

(d) Using part (a) and the Inverse Function Theorem, show that if $(f, \mathbf{z}) \in V - \Sigma'$ then there exists an open neighborhood of $(f, \mathbf{z})$ where the inverse function of π_V exists, and is continuously differentiable. Here $\Sigma' = \{(f, \mathbf{z}) \in V \ : \ \mathrm{rank}(Df(\mathbf{z})) < n\}$.

(c) Let $f^* = (x_1^{d_1} - x_0^{d_1}, x_2^{d_2} - x_0^{d_2}, \ldots, x_n^{d_n} - x_0^{d_n}) \in \mathcal{H}_{(d)}$ and denote by $\xi_1, \ldots, \xi_{\mathcal{D}}$ its $\mathcal{D} = \prod_{i=1}^n d_i$ distinct common roots. Show that for all $\varepsilon > 0$ there exists a neighborhood U of f^* such that for all $f \in U$, f has $\mathcal{D}$ distinct roots $\zeta_1, \ldots, \zeta_{\mathcal{D}}$ such that $\|\xi_i - \zeta_i\| < \varepsilon$ for all $i = 1, \ldots, \mathcal{D}$.

3. **Newton's method in projective space**

Let $f = (f_1, \ldots, f_n)$ be a vector of n homogeneous polynomials of degrees $(d) = (d_1, \ldots, d_k)$ in the variables $\mathbf{x} = (x_0, x_1, \ldots, x_n)$. Recall that for $\mathbf{z} \in \mathbb{C}^{n+1}$

$$T_{\mathbf{z}} = \{\mathbf{w} \in \mathbb{C}^{n+1} \ : \ \mathbf{z} \cdot \overline{\mathbf{w}} = 0\} \subset \mathbb{C}^{n+1}$$

is the tangent space of $\mathbb{P}(\mathbb{C}^{n+1})$ at $\mathbf{z}$. Also recall that the derivative of f at $\mathbf{z}$ is a linear map $Df(\mathbf{z}) : \mathbb{C}^{n+1} \to \mathbb{C}^n$ defined by the Jacobian matrix of f. Denote by $Df(\mathbf{z})|_{T_{\mathbf{z}}}$ the restriction of $Df(\mathbf{z})$ to $T_{\mathbf{z}}$, and by $Df(\mathbf{z})|_{T_{\mathbf{z}}}^{-1}$ its inverse (assume it exists). Define the Newton step by

$$N_f(\mathbf{z}) = \mathbf{z} - Df(\mathbf{z})|_{T_{\mathbf{z}}}^{-1} f(\mathbf{z})$$

(a) Prove that the map N_f takes the line through $\mathbf{z}$ and 0 into a line through $N_f(\mathbf{z})$ and 0.
(b) Prove that $N_f(\mathbf{z}) = \mathbf{z}$ if and only if $f(\mathbf{z}) = 0$.
(c) Implement in Maple the Newton iteration for $n = 2$.
(d) Run the Newton iteration with data $f = x^2 + y^2 + z^2$, $g = 2xy - xz + z^2$ and starting point $\mathbf{z}_0 := (I, (1+I)/2, 1)$ where $I = \sqrt{-1}$. Does it converge?