Multivariate subresultants using Jouanolou's resultant matrices

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Abstract

Earlier results expressing multivariate subresultants as ratios of two subdeterminants of the Macaulay matrix are extended to Jouanolou's resultant matrices. These matrix constructions are generalizations of the classical Macaulay matrices and involve matrices of significantly smaller size. Equivalence of the various subresultant constructions is proved. The resulting subresultant method improves the efficiency of previous methods to compute the solution of over-determined polynomial systems.

Key words: Multivariate subresultants, Jouanolou's resultant matrix, Koszul-Weyman complex, solution of polynomial systems

1 Introduction

The primary concern of the present paper is to find efficient methods to compute multivariate generalizations of the univariate subresultants. Univariate subresultants were introduced originally by Sylvester [14] and rediscovered by Collins in [5] where subresultants were used to give an efficient and parallelizable algorithm

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[1] Research supported in part by EPRSC while a postdoctoral fellow at the University of Kent, Canterbury, and in part by the support of NSF grants CCR-0306406 and CCR-0347506.

to compute the greatest common divisor of two univariate polynomials. Multivariate subresultants generalize the classical univariate subresultants in the sense that they provide the coefficients of certain polynomials which in the univariate case include the greatest common divisor of two given polynomials. González Vega in [9,10] gives a multivariate generalization of the univariate subresultant method using a non-homogeneous construction by Habicht [11]. He defines the subresultants as subdeterminants of the Macaulay matrix, and then constructs a geometric representation of the zero-dimensional solution set of a given polynomial system using them. Chardin [3,4] introduces a more general version of subresultants as the ratio of two subdeterminants of the Macaulay matrix, and proves that they satisfy some universal properties, described below in the preliminaries.

In this paper we define subresultants using a generalization of the Macaulay matrix, a matrix construction introduced by Jouanolou in [12]. The entries of this matrix include coefficients of the given polynomials and their so called Morley forms, described below. We prove that our construction gives the same subresultants as the Macaulay type constructions [3]. The practical advantage of using our matrix constructions is that the size of the matrices is smaller than in the constructions using Macaulay matrices. The resulting method improves the efficiency of the solution of over-determined polynomial systems, which is the subject of the paper [16]. On a more theoretical level, we believe that our general formulation of subresultants gives an understanding of the connection between Koszul complexes in different degrees, bringing us closer to an understanding of the connection between the geometric and the algebraic structure of the solution of non-generic polynomial systems.

The paper is structured as follows.

- In the preliminaries, after recalling the univariate subresultant construction of Collins [5], we describe multivariate subresultants using Macaulay's matrices defined by González Vega and Chardin [3,4,9,10]. We then describe Jouanolou's resultant matrix construction [12].
- Section 3 contains the description of the subresultant construction based on Jouanolou's resultant matrices.
  - In subsection 3.1, we give the constructions for the submatrices of Jouanolou's resultant matrices which we later use in the definition of the subresultants. We prove that these submatrices have generically maximal rank.
  - In subsection 3.2 we define the subresultants as the ratios of two minors of the resultant matrices of Jouanolou. We prove that the subresultants are polynomials in the coefficients of the given polynomial system of the same degree as the subresultants constructed from Macaulay's matrix. Furthermore, in this subsection we prove that the non-vanishing of a particular subresultant

is equivalent to that certain polynomials with given support are in the ideal generated by the given polynomials.

- In subsection 3.3 we describe the subresultants as the determinants of certain Koszul-Weyman type complexes. This construction is needed in order to prove the main theorem of the paper, that the Jouanolou type subresultants are the same as the Macaulay type subresultants. The proof involves the understanding of non-exactness of Koszul type complexes in a fixed degree and its connection to the non-exactness of Koszul type complexes in a different degree.

We note here that an anonymous referee suggested an alternative way to present the results of the paper: Define the Jouanolou type subresultants as determinants of based Koszul-Weyman type complexes (see Definitions 3.3.1 and 3.3.4). Then prove that the subresultants defined this way are the same as the subresultants defined in [4] using Macaulay matrices (see Theorem 3.3.10). Finally, deduce the formula for the subresultant as the ratio of two determinants as in Definition 3.2.1. The advantage of this presentation could be that some of the properties of the Jouanolou type subresultants proved here could be derived directly from those already proved for the Macaulay type subresultants in [4]. However, it is not clear how to prove Theorem 3.3.10 without using these properties of the Jouanolou type subresultants (e.g. degrees). Thus the suggested alternative presentation may not simplify or reduce the length of the paper, so we kept the original presentation.

2 Preliminaries

2.1 Subresultants à la Macaulay

Before we describe the mutivariate constructions of González-Vega [9] and Chardin [4], let us recall the classical univariate subresultant construction (cf. [5] or [10]).

Let $f_1 = \sum_{i=0}^{d_1} a_i x^i$ and $f_2 = \sum_{i=0}^{d_2} b_i x^i$ be two univariate polynomials of degree $\deg(f_1) = d_1$ and $\deg(f_2) = d_2$ with coefficients from an integral domain $\mathbf{R}$ which has quotient field $\mathbf{K}$. For each $i = 0, \dots, \min(d_1, d_2) - 1$ we can define the matrix

$$\mathbf{S}^i := \begin{array}{cl} \quad d_1 + d_2 - i & \\ \left|\begin{array}{cccccc} a_0 & \dots & a_{d_1} & & & \\ & \ddots & & & \ddots & \\ & & & a_0 & \dots & a_{d_1} \\ b_0 & \dots & b_{d_2} & & & \\ & \ddots & & & \ddots & \\ & & & b_0 & \dots & b_{d_2} \end{array}\right| & \begin{array}{l} \\ d_2 - i \\ \\ \\ d_1 - i \\ \\ \end{array} \end{array}$$

with rows corresponding to the polynomials $x^j \cdot f_1$ $(0 \leq j < d_2 - i)$ and $x^j \cdot f_2$ $(0 \leq j < d_1 - i)$. Note that $\mathbf{S}^0$ is the Sylvester matrix of f_1, f_2, and $\mathbf{S}^i$ is a submatrix of $\mathbf{S}^0$ obtained by deleting $2i$ rows and i columns.

Assume that $i \geq 1$. For any $0 \leq j \leq i$ we can define $\mathbf{S}^i_j$ to be the square submatrix of $\mathbf{S}^i$ obtained by removing the columns indexed by the set $\{1, 2, \dots, i+1\} - \{j+1\}$. The *scalar subresultant* Δ^i_j of (f_1, f_2) is defined by

$$\Delta^i_j := \det(\mathbf{S}^i_j). \tag{1}$$

Note that classically univariate subresultants are defined as polynomials in x with coefficients the scalar subresultants defined above (see (2) below). The reason we gave the definition of scalar subresultants is that they generalize to the notion we use for multivariate subresultants.

Assume that $\deg(f_1) \geq \deg(f_2)$ and the leading coefficient of f_1 is non-zero, i.e. $a_{d_1} \neq 0$. Then the following statements hold (cf. [5,10]):

(1) The greatest common divisor of f_1 and f_2 in $\mathbf{K}[x]$ has degree i if and only if

$$\det(\mathbf{S}^0) = \Delta^1_1 = \cdots = \Delta^{i-1}_{i-1} = 0 \text{ and } \Delta^i_i \neq 0.$$

(2) For each $i = 0, \dots, \min(d_1, d_2) - 1$ the polynomials

$$\sum_{j=0}^{i} \Delta^i_j \cdot x^j = \Delta^i_0 + \Delta^i_1 \cdot x + \cdots \Delta^i_i \cdot x^i \tag{2}$$

are in the ideal $\langle f_1, f_2 \rangle \subset \mathbf{R}[x]$. In particular, if $\gcd_{\mathbf{K}[x]}(f_1, f_2)$ has degree i than it is equal to $\sum_{j=0}^{i} \Delta^i_j \cdot x^j$ in $\mathbf{K}[x]$.

For homogeneous multivariate polynomials systems González-Vega [9] and Chardin [4] generalized the notion of univariate subresultants. Let us recall the properties of the multivariate subresultant construction following the approach in [4]. Let

$$f_1 = \sum_{|\alpha|=d_1} c_{1,\alpha} x^\alpha, \ldots, f_s = \sum_{|\alpha|=d_s} c_{s,\alpha} x^\alpha \in \mathbf{R}[x_1, \ldots, x_n]$$

be homogeneous polynomials with degrees $d = (d_1, \ldots d_s)$ and with parametric coefficients $c_{i,\alpha}$ where $\mathbf{R}$ is a Noetherian UFD containing $\mathbb{Z}[c_{i,\alpha}]$. To simplify the notation x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Given $\nu \in \mathbb{N}$, let $S \subseteq \mathrm{Mon}(\nu)$ be a set of monomials of degree ν. Assume that S has cardinality $\mathcal{H}_d(\nu)$, where $\mathcal{H}_d$ denotes the Hilbert function of a regular sequence of s polynomials with degrees $d = (d_1, \ldots, d_s)$ (see e.g. [4]). Moreover, assume that

$$\mathbf{K}\langle S \rangle + I_\nu = \mathbf{K}[x_1, \ldots, x_n]_\nu$$

where $\mathbf{K}$ is the fraction field of $\mathbf{R}$ and I_ν denotes the degree ν part of the ideal $\langle f_1, \ldots, f_s \rangle_{\mathbf{K}}$. Then Chardin in [4] defines the polynomials $\Delta_S^\nu(f) \in \mathbb{Z}[c_{i,\alpha}]$ satisfying the following properties:

(1) If $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_s) \in k[x_0, \ldots, x_n]$ are coefficient specializations of the polynomials $f = (f_1, \ldots, f_s)$ (k is a field) then

$$\Delta_S^\nu(\tilde{f}) \neq 0 \quad \text{if and only if} \quad \tilde{I}_\nu + k\langle S \rangle = k[x_0, \ldots, x_n]_\nu.$$

Here $\tilde{I}_\nu$ denotes the degree ν part of the ideal $\langle \tilde{f}_1, \ldots, \tilde{f}_s \rangle$.

(2) For any fixed $1 \leq i \leq s$, Δ_S^ν is a homogeneous polynomial in the coefficients $c_{i,\alpha}$ ($|\alpha| = d_i$) of degree $\mathcal{H}_{\hat{d}^i}(\nu - d_i)$. Here $\mathcal{H}_{\hat{d}^i}$ denotes the Hilbert function of a regular sequence of $s-1$ homogeneous polynomials in n variables with degrees $\hat{d}^i = (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_s)$.

(3) For any $x^\alpha \notin S$ of degree ν we have

$$\Delta_S^\nu \cdot x^\alpha + \sum_{x^\beta \in S} \epsilon_\beta \cdot \Delta^\nu_{(S \cup \{x^\alpha\} - \{x^\beta\})} \cdot x^\beta \quad \in \langle f_1, \ldots, f_n \rangle_\nu \tag{3}$$

where $\epsilon_\beta = \pm 1$.

(4) In the case when $s = n$ and $\nu > \sum_{i=1}^n (d_i - 1)$, we have $\mathcal{H}_d(\nu) = 0$ and $\Delta_\emptyset^\nu = \mathrm{Res}_d(f)$, the projective-resultant (see next subsection for definition).

Note that 1. and 2. are universal properties in the sense that the subresultant Δ_S^ν is determined by them up to a constant multiple. In the special case when $n = 2$, for $0 \leq i \leq \min(d_1, d_2) - 1$ and $0 \leq j \leq i$, the univariate subresultant Δ_j^i defined in (1) is the same as Δ_S^ν for $\nu = d_1 + d_2 - i$ and $S = \{x_1^\nu,\ x_1^{\nu-1}x_2,\ \ldots,\ x_1^{\nu-i}x_2^i\} - \{\ x_1^{\nu-j}x_2^j\ \}$.

For the case when $s = n$, the subresultant construction of González Vega [9,10] is defined as generating polynomials with fixed pattern in the ideal generated by $f_1, \ldots, f_n$, using subdeterminants of the Macaulay matrix. His definition is analogous to the notion and construction of classical univariate subresultants. In [4] Chardin defines multivariate subresultants as Δ_S^ν, and constructs them as the determinants of the degree ν homogeneous part of the Koszul complex of $f_1, \ldots, f_s$ restricted to $\langle \mathrm{Mon}(\nu) - S \rangle$. This is an alternating product of subdeterminants corresponding to matrices of the differentials of the Koszul complex. Finally, Chardin in [3] expresses Δ_S^ν as the ratio of two subdeterminants of the Macaulay matrix.

Example 2.1.1 For $n = 3$ consider 3 generic polynomials $f = (f_1, f_2, f_3)$ in the variables (x, y, z) of degrees $d = (3, 3, 2)$:

$$\begin{aligned} f_1 =& a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3 \\ f_2 =& b_0x^3 + b_1x^2y + b_2x^2z + b_3xy^2 + b_4xyz + b_5xz^2 + b_6y^3 + b_7y^2z + b_8yz^2 + b_9z^3 \\ f_3 =& c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2. \end{aligned} \tag{4}$$

Taking $\nu = 5$ the submatrix of the Macaulay matrix corresponding to the subresultant has size 20×21 and we do not include it here. Taking $\nu = 4$, the Macaulay type subresultant matrix is the following 12×15 matrix

$$\mathbf{M} := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & a_1 & a_2 & 0 & a_3 & a_4 & a_5 & 0 & a_6 & a_7 & a_8 & a_9 & 0 \\ 0 & 0 & a_0 & 0 & a_1 & a_2 & 0 & a_3 & a_4 & a_5 & 0 & a_6 & a_7 & a_8 & a_9 \\ c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_0 & 0 & b_1 & b_2 & 0 & b_3 & b_4 & b_5 & 0 & b_6 & b_7 & b_8 & b_9 & 0 \\ 0 & 0 & b_0 & 0 & b_1 & b_2 & 0 & b_3 & b_4 & b_5 & 0 & b_6 & b_7 & b_8 & b_9 \\ 0 & 0 & 0 & c_0 & 0 & 0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & 0 & 0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & c_5 \end{bmatrix}$$

with rows corresponding to monomials

$$\begin{bmatrix} x^4 & x^3y & x^3z & z^2x^2 & y^3x & z^2yx & z^3x & y^4 & y^3z & z^2y^2 & z^3y & z^4 \end{bmatrix},$$

and columns corresponding to monomials

$$\begin{bmatrix} x^4 & x^3y & x^3z & y^2x^2 & yx^2z & z^2x^2 & y^3x & y^2xz & z^2yx & z^3x & y^4 & y^3z & z^2y^2 & z^3y & z^4 \end{bmatrix}.$$

Taking any $S \subset \mathrm{Mon}(4)$ of cardinality $\mathcal{H}_{(3,3,2)}(4) = 3$, the columns of $\mathbf{M}$ not corresponding to S form a square matrix $\mathbf{M}_S$. In this example the determinant of $\mathbf{M}_S$ is equal to the subresultant Δ_S^ν. (Note that in general the subresultant Δ_S^ν is a ratio of two subdeterminants.) $\boxdot$

In this section we recall the definition of projective resultants and describe the construction of Jouanolou for Morley forms and resultant matrices (cf. [12, Section 3.10]).

First we give the definition of *projective-resultants.*

Definition 2.2.1 *Let*

$$f_1 = \sum_{|\alpha|=d_1} c_{1,\alpha} x^\alpha, \; \ldots, \; f_n = \sum_{|\alpha|=d_n} c_{n,\alpha} x^\alpha$$

be "generic" homogeneous polynomials of degrees $d = (d_1, \ldots d_n)$, i.e. the coefficients $c_{i,\alpha}$ are parameters, and we consider $f_1, \ldots, f_n$ as polynomials in the ring $\mathbb{Z}[c_{i,\alpha} : |\alpha| = d_i, 1 \leq i \leq n][x_1, \ldots, x_n]$. Then there exists a polynomial Res_d such that Res_d is an irreducible element of $\mathbb{Z}[c_{i,\alpha}]$ depending only on the degrees $d = (d_1, \ldots, d_n)$, and for any complex coefficient specialization $\tilde{f}_1, \ldots, \tilde{f}_n \in \mathbb{C}[x_1, \ldots, x_n]$ of $f_1, \ldots, f_n$ we have

$$\{x \in \mathbb{P}^{n-1}_{\mathbb{C}} \mid \tilde{f}_1(x) = \cdots \tilde{f}_n(x) = 0\} \neq \emptyset \;\; \Leftrightarrow \;\; \mathrm{Res}_d(\tilde{f}_1, \ldots, \tilde{f}_n) = 0.$$

Res_d is called the projective-resultant in degrees $d = (d_1, \ldots, d_n)$. For proofs and a more general definition of resultants we refer to [8]. Note that the above results remain true if we replace the complex field $\mathbb{C}$ by any algebraically closed field of characteristic zero.

In order to define Jouanolou's matrix construction for the projective-resultant let us first fix the notation. Let $f_1, \ldots, f_n$ be homogeneous polynomials in $\mathbf{R}[x_1, \ldots, x_n]$ with degrees $d = (d_1, \ldots d_n)$ where $\mathbf{R}$ is a Noetherian UFD. Denote by δ the sum

$$\delta = \sum_{i=1}^{n} (d_i - 1).$$

Definition 2.2.2 *Let $d = (d_1, \ldots d_n)$ be as above. For $\eta \geq 0$ we define the following sets of monomials*

$$\begin{aligned}
\mathrm{Mon}_n(\eta) &:= \{x^\alpha \mid |\alpha| = \eta\} \\
\mathrm{Rep}_d(\eta) &:= \{x^\alpha \mid |\alpha| = \eta, \; \exists i \; \alpha_i \geq d_i\} \\
\mathrm{Dod}_d(\eta) &:= \{x^\alpha \mid |\alpha| = \eta, \; \exists i \neq j \; \alpha_i \geq d_i, \alpha_j \geq d_j\}.
\end{aligned}$$

The notations Mon, Rep_d *and* Dod_d *are borrowed from [12] and stand for monômes, d-repus and d-dodus, respectively. We may omit to note n if it is clear from the context. Also, we denote by* $\mathrm{Mon}^*(\eta)$ *the dual basis of* $\mathrm{Mon}(\eta)$ *in the dual* **R**-*module* $\langle \mathrm{Mon}(\eta)\rangle^*$, *and similarly for* $\mathrm{Rep}_d^*(\eta)$. *For $\eta < 0$ we define all of the above sets to be the empty set.*

Next we define the *Morley forms.*

Definition 2.2.3 *Let $y = (y_1, \ldots, y_n)$ be a new set of variables. For each $1 \le i, j \le n$ we define the discrete differentials $\theta_{i,j}$ by*

$$\theta_{i,j}(x, y) := \frac{f_i(y_1, \ldots y_{j-1}, x_j, \ldots, x_n) - f_i(y_1, \ldots y_{j-1}, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}.$$

The determinant of $(\theta_{i,j})_{1\le i,j\le n}$ is called the Bezoutian.
Note that our definition of Bezoutians is different from the Bezoutians defined in [1,2], which is defined for n non-homogeneous polynomials in $n-1$ variables, and is in the ideal generated by the polynomials.

We use the term Morley form to denote the coefficient Morl_γ *of y^γ in the Bezoutian, i.e. we have*

$$\det(\theta_{i,j})_{1\le i,j\le n} = \sum_{|\gamma|\le\delta} \mathrm{Morl}_\gamma(x) y^\gamma. \qquad (5)$$

Note that the degree of $\mathrm{Morl}_\gamma(x)$ *is* $\delta - |\gamma|$.

Next we define the resultant matrices of Jouanolou. We have the following remark first.

Remark 2.2.4 Throughout this paper we chose to use the same notation for linear maps and their matrices in the bases the maps were defined in. Since all linear maps in the paper are defined for fixed bases, and we do not change these bases throughout the paper, this abuse of notation will not lead to ambiguity. Also, in our matrix notation, each row corresponds to an element of the basis of the domain and each column correspond to an element of the basis of the image space, thus the matrices are acting on the right hand side. Throughout the paper the dual of a linear map ϕ is denoted by ϕ^*, therefore the transpose of the matrix of ϕ, corresponding to the map ϕ^*, is also denoted by ϕ^*.

Definition 2.2.5 *For any fixed $0 \le \eta \le \delta + 1$ the Jouanolou resultant matrix*

$\mathbf{J}_\eta(f)$ *has the following structure:*

$$\mathbf{J}_\eta(f) = \left[\begin{array}{c|c} \Omega_\eta & \Phi_\eta^* \\ \hline \Phi_{\delta-\eta} & 0 \end{array}\right].$$

The blocks of the matrix $\mathbf{J}_\eta(f)$ *correspond to the following* $\mathbf{R}$*-linear maps:*

For $0 \leq t \leq \delta$ *define*

$$\Omega_t : \langle \mathrm{Mon}(t) \rangle^* \to \langle \mathrm{Mon}(\delta - t) \rangle, \qquad y^\beta \mapsto \mathrm{Morl}_\beta(x). \tag{6}$$

If $x^\alpha \in \mathrm{Rep}_d(t)$ *then let* $i(\alpha)$ *be the smallest index such that* $\alpha_{i(\alpha)} \geq d_{i(\alpha)}$ *and define*

$$\Phi_t : \langle \mathrm{Rep}_d(t) \rangle \to \langle \mathrm{Mon}(t) \rangle, \qquad x^\alpha \mapsto \left(\frac{x^\alpha}{x_{i(\alpha)}^{d_{i(\alpha)}}} \right) \cdot f_{i(\alpha)}. \tag{7}$$

The dual of the $\mathbf{R}$*-linear map* Φ_η *is denoted by*

$$\Phi_\eta^* : \langle \mathrm{Mon}(\eta) \rangle^* \to \langle \mathrm{Rep}_d(\eta) \rangle^*, \qquad y^\beta \mapsto \sum_{x^\alpha \in \mathrm{Rep}_d(\eta)} y^\beta \left(\Phi_\eta(x^\alpha) \right) \cdot y^\alpha \tag{8}$$

where $y^\beta(x^\gamma) = \delta_{\beta,\gamma}$*. Finally, the matrix* $\mathbf{J}_\eta(f)$ *corresponds to the following* $\mathbf{R}$*-linear map, also denoted by* $\mathbf{J}_\eta(f)$*:*

$$\begin{aligned} \mathbf{J}_\eta(f) : \langle \mathrm{Mon}(\eta) \rangle^* \oplus \langle \mathrm{Rep}_d(\delta - \eta) \rangle &\to \langle \mathrm{Mon}(\delta - \eta) \rangle \oplus \langle \mathrm{Rep}_d(\eta) \rangle^* \\ \left(y^\beta, x^\alpha \right) &\mapsto \left(\Omega_\eta(y^\beta) + \Phi_{\delta-\eta}(x^\alpha), \;\; \Phi_\eta^*(y^\beta) \right) \end{aligned} \tag{9}$$

for $y^\beta \in \mathrm{Mon}^*(\eta)$ *and* $x^\alpha \in \mathrm{Rep}_d(\delta - \eta)$*.*

Theorem 2.2.6 *[6,12]*
Let $f = (f_1, \ldots, f_n)$ *be generic homogeneous polynomials in* $\mathbf{R}[x_1, \ldots, x_n]$ *of degree* $d = (d_1, \ldots, d_n)$*. Then*

(1) For all $0 \leq \eta \leq \delta + 1$*, Jouanolou's matrix* $\mathbf{J}_\eta(f)$ *is square. [6,12]*

(2) For all $0 \le \eta \le \delta+1$

$$\mathrm{Res}_d(f) = \frac{\det(\mathbf{J}_\eta(f))}{\det(\mathbf{E}_{\delta-\eta}(f))\det(\mathbf{E}_\eta(f))}$$

where $\mathbf{E}_\eta(f)$ ($\mathbf{E}_{\delta-\eta}(f)$, *resp.) is the submatrix of the matrix* $\mathbf{J}_\eta(f)$ *with rows and columns corresponding to monomials in* $\mathrm{Dod}_d(\eta)$ ($\mathrm{Dod}_d(\delta-\eta)$, *resp.). [6]*

(3) For all $0 \le s, t \le \delta+1$ *and for* $x^\alpha \in \mathrm{Mon}(s)$

$$x^\alpha \sum_{y^\beta \in \mathrm{Mon}^*(t)} y^\beta \mathrm{Morl}_\beta(x) - y^\alpha \sum_{y^\gamma \in \mathrm{Mon}^*(t-s)} y^\gamma \mathrm{Morl}_\gamma(x) \tag{10}$$

is in the ideal $\langle f_1(x), \ldots, f_n(x), f_1(y), \ldots, f_n(y)\rangle$ *[12, 3.11.11]).*

(4) Let C *denote the matrix corresponding to the map* $\Phi_\eta^* : \langle \mathrm{Mon}(\eta)\rangle^* \to \langle \mathrm{Rep}_d(\eta)\rangle^*$ *and let* B *denote the column vector* $(x^\gamma \mathrm{Morl}_\beta(x))_{|\beta|=\eta}$ *where* x^γ *is any fixed element of* $\mathrm{Mon}(\eta+1)$*. Then any maximal minor of the matrix*

$$\begin{array}{cl} \begin{array}{|c|c|} \hline & \\ B & C \\ & \\ \hline \end{array} & \#\mathrm{Mon}(\eta) \\ \#\mathrm{Rep}_d(\eta)+1 & \end{array} \tag{11}$$

is in the ideal $\langle f_1(x), \ldots, f_n(x)\rangle$*. [12, Proposition 3.11.19.3]*

Example 2.1.1 (cont)
Let $n = 3$, $d = (3,3,2)$ and $f = (f_1, f_2, f_3)$ be polynomials in $\mathbf{x} := (x, y, z)$ as in Example 2.1.1, i.e.

$$\begin{aligned} f_1 =& a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3 \\ f_2 =& b_0x^3 + b_1x^2y + b_2x^2z + b_3xy^2 + b_4xyz + b_5xz^2 + b_6y^3 + b_7y^2z + b_8yz^2 + b_9z^3 \\ f_3 =& c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2. \end{aligned} \tag{12}$$

Using the variables $\mathbf{u} := (u, v, w)$ for the dual $\mathbf{R}$-algebra, the discrete differentials $\theta_{i,j}$ all have similar forms as the following instance:

$$\theta_{1,2} = a_6y^2 + a_7yw + a_3xy + a_6yv + a_1x^2 + a_8w^2 + a_4xw + a_7vw + a_3xv + a_6v^2.$$

The determinant of the matrix $(\theta_{i,j})$ is the Bezoutian, we cannot include it here. The Morley forms – coefficients of the Bezoutian as a polynomial in u, v, w – have multilinear coefficients. For example Morl_{uv} have coefficients like this one:

$$\mathrm{Morl}_{uv} = \cdots + (-a_1c_1b_5 + a_3c_0b_5 - a_0c_3b_5 + a_5b_1c_1 + a_5b_0c_3 - a_3b_0c_5 - a_5c_0b_3 + a_0b_3c_5)x^2y + \cdots.$$

Note that the above coefficient is a "bracket polynomial", i.e. it is a 3×3 subdeterminant of the coefficient matrix of f.

Jouanolou's matrix $\mathbf{J_2}(f)$ for $\eta = 2$ is the following 11×11 matrix:

$$\begin{bmatrix}
\mu_{u^2,x^3} & \mu_{u^2,x^2y} & \mu_{u^2,x^2z} & \mu_{u^2,xy^2} & \mu_{u^2,xyz} & \mu_{u^2,xz^2} & \mu_{u^2,y^3} & \mu_{u^2,y^2z} & \mu_{u^2,yz^2} & \mu_{u^2,z^3} & c_0 \\
\mu_{vu,x^3} & \mu_{vu,x^2y} & \mu_{vu,x^2z} & \mu_{vu,xy^2} & \mu_{vu,xyz} & \mu_{vu,xz^2} & \mu_{vu,y^3} & \mu_{vu,y^2z} & \mu_{vu,yz^2} & \mu_{vu,z^3} & c_1 \\
\mu_{wu,x^3} & \mu_{wu,x^2y} & \mu_{wu,x^2z} & \mu_{wu,xy^2} & \mu_{wu,xyz} & \mu_{wu,xz^2} & \mu_{wu,y^3} & \mu_{wu,y^2z} & \mu_{wu,yz^2} & \mu_{wu,z^3} & c_2 \\
\mu_{v^2,x^3} & \mu_{v^2,x^2y} & \mu_{v^2,x^2z} & \mu_{v^2,xy^2} & \mu_{v^2,xyz} & \mu_{v^2,xz^2} & \mu_{v^2,y^3} & \mu_{v^2,y^2z} & \mu_{v^2,yz^2} & \mu_{v^2,z^3} & c_3 \\
\mu_{wv,x^3} & \mu_{wv,x^2y} & \mu_{wv,x^2z} & \mu_{wv,xy^2} & \mu_{wv,xyz} & \mu_{wv,xz^2} & \mu_{wv,y^3} & \mu_{wv,y^2z} & \mu_{wv,yz^2} & \mu_{wv,z^3} & c_4 \\
\mu_{w^2,x^3} & \mu_{w^2,x^2y} & \mu_{w^2,x^2z} & \mu_{w^2,xy^2} & \mu_{w^2,xyz} & \mu_{w^2,xz^2} & \mu_{w^2,y^3} & \mu_{w^2,y^2z} & \mu_{w^2,yz^2} & \mu_{w^2,z^3} & c_5 \\
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & 0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & 0 & 0 & 0 & 0 & 0 \\
b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & 0 \\
0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0 & 0 \\
0 & 0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0
\end{bmatrix} \qquad (13)$$

where $\mu_{\mathbf{u}^\beta,\mathbf{x}^\alpha}$, denotes the coefficient of $\mathbf{x}^\alpha$ in $\mathrm{Morl}_{\mathbf{u}^\beta}(\mathbf{x})$.
The rows of the resultant matrix correspond to the monomials

$$\begin{bmatrix} u^2 & uv & uw & v^2 & vw & w^2 & x^3 & z^2x & y^3 & z^2y & z^3 \end{bmatrix}$$

and the columns correspond to the monomials

$$\begin{bmatrix} x^3 & yx^2 & zx^2 & y^2x & zyx & z^2x & y^3 & y^2z & z^2y & z^3 & w^2 \end{bmatrix}.$$

Since $\mathrm{Dod}_{(3,3,2)}(2) = \mathrm{Dod}_{(3,3,2)}(3) = \emptyset$, the determinant of Jouanolou's matrix is the resultant.

Note that Macaulay's resultant matrix (which is a special case of Jouanolou's matrices for $\eta = \delta + 1 = 6$) has size 28×28, which we do not include here. Its determinant is a nontrivial multiple of the resultant. The ratio of the determinant and the resultant is the determinant of the following matrix $\mathbf{E}_6$:

$$\begin{bmatrix}
a_0 & 0 & 0 & 0 & b_0 & 0 & 0 \\
0 & a_0 & 0 & 0 & 0 & 0 & 0 \\
a_1 & 0 & a_0 & 0 & b_1 & b_0 & 0 \\
a_2 & 0 & 0 & a_0 & b_2 & 0 & b_0 \\
a_6 & a_5 & a_3 & 0 & b_6 & b_3 & 0 \\
0 & a_8 & a_6 & 0 & 0 & b_6 & 0 \\
0 & a_9 & a_7 & a_6 & 0 & b_7 & b_6
\end{bmatrix}$$

with rows and columns corresponding to the monomials

$$\mathrm{Dod}_{(3,3,2)}(6) = \begin{bmatrix} x^4z^2 & x^3y^3 & x^3yz^2 & x^3z^3 & y^3xz^2 & y^4z^2 & y^3z^3 \end{bmatrix} \quad \boxdot$$

3 Subresultants à la Jouanolou

Let $f_1 = \sum_{|\alpha|=d_1} c_{1,\alpha} x^\alpha, \ \ldots, \ f_n = \sum_{|\alpha|=d_n} c_{n,\alpha} x^\alpha$ be generic homogeneous polynomials in $\mathbf{R}[x_1, \ldots, x_n]$ with degrees $d = (d_1, \ldots d_n)$ where $\mathbf{R}$ is a Noetherian UFD containing a field k of characteristic zero and $\mathbb{Z}[c_{i,\alpha}]$. Denote by δ the sum $\sum_{i=1}^n (d_i - 1)$ as before, and fix $\eta \geq 0$ and $\nu \geq 0$.

In this section we define a matrix $\mathbf{J}_{\eta,\nu}(f)$, a submatrix of Jouanolou's matrix $\mathbf{J}_\eta(f)$ defined in (9), such that it gives an analogue to the Macaulay type subresultant of degree ν. The motivation for the otherwise arbitrary construction of $\mathbf{J}_{\eta,\nu}(f)$ is to obtain a submatrix of $\mathbf{J}_\eta(f)$ which has the following properties:

(1) The difference between the number of rows and columns of $\mathbf{J}_{\eta,\nu}(f)$ is $\mathcal{H}_d(\nu)$, the same as the difference between the number of rows and columns of the Macaulay type subresultant matrix of degree ν (see [4]).
(2) There exists submatrices $\mathbf{E_1}, \mathbf{E_2}$ of $\mathbf{J}_{\eta,\nu}(f)$ such that any maximal minor of $\mathbf{J}_{\eta,\nu}(f)$ divided by $\det(\mathbf{E_1}) \cdot \det(\mathbf{E_2})$ is a homogeneous polynomial in the coefficients of f_i of degree $\mathcal{H}_{\hat{d^i}}(\nu - d_i)$ which is the same as the degree of the Macaulay type subresultant. These homogeneous polynomials are going to be our subresultants.
(3) The non-vanishing of a particular subresultant is equivalent to that $f_1, \ldots, f_n$ 'pseudo-generates' all monomials of degree $\delta - \eta$, except maybe a particular subset of cardinality $\mathcal{H}_d(\nu)$. (See Proposition 3.2.4 and Lemma 3.3.9 for the meaning of the term 'pseudo-generate'.)

3.1 Construction of the subresultant matrix

First we define sets of monomials corresponding to columns and rows of Jouanolou's resultant matrix $\mathbf{J}_\eta(f)$ to be removed to obtain the submatrices $\mathbf{J}_{\eta,\nu}(f)$.

Definition 3.1.1 *Fix $d = (d_1, \ldots, d_n)$. For $0 \leq q \leq p$ let*

$$\begin{aligned} \overline{\mathrm{Mon}}_n(p,q) &:= \{x^\alpha \mid |\alpha| = p, \alpha_n \geq q\} \\ \overline{\mathrm{Rep}}_d(p,q) &:= \{x^\alpha \in \overline{\mathrm{Mon}}_n(p,q) \mid \exists i \leq n-1 \ \ \alpha_i \geq d_i \ \ \text{or} \ \ \alpha_n \geq d_n + q\} \end{aligned}$$

Note that there are bijections between the sets

$$\overline{\mathrm{Mon}}_n(p,q) \cong \mathrm{Mon}_n(p-q) \ \ \text{and} \ \ \overline{\mathrm{Rep}}_d(p,q) \cong \mathrm{Rep}_d(p-q)$$

by taking $\alpha'_n := \alpha_n - q$ (see also Definition 2.2.2). We denote the sets of monomials corresponding to columns and rows of $\mathbf{J}_{\eta,\nu}(f)$ by

$$\begin{aligned}\mathrm{Mon}_n(p,q) &:= \mathrm{Mon}_n(p) - \overline{\mathrm{Mon}}_n(p,q)\\ \mathrm{Rep}_d(p,q) &:= \mathrm{Rep}_d(p) - \overline{\mathrm{Rep}}_d(p,q).\end{aligned}$$

We may omit to note n if it is clear from the context. We also define here the set

$$H_d(t) := \{x^\alpha \mid |\alpha| = t, \forall i \;\; \alpha_i < d_i\}$$

which has cardinality $\mathcal{H}_d(t)$ (as before, $\mathcal{H}_d$ denotes the Hilbert function of a regular sequence of n polynomials with degrees $d = (d_1, \ldots, d_n)$).

As before let $\delta = \sum_{i=1}^n (d_i - 1)$. Fix η and ν such that they satisfy the condition

$$0 \le \delta - \nu \le \eta \le \delta - \eta \le \nu \le \delta. \tag{14}$$

Informally, η denotes the smaller one among η and $\delta - \eta$ in the definition of Jouanolou's matrix and ν is the analogue of the degree in the Macaulay type subresultant construction. Assumption (14) ensures that we remove rows only from the submatrices Ω_η and Φ^*_η of $\mathbf{J}_\eta$.

To simplify the notation we denote $\eta' := \eta - (\delta - \nu)$. Using Definitions 3.1.1 we give explicitly the sets

$$\begin{aligned}\mathrm{Mon}(\eta,\eta') &= \{x^\alpha \mid |\alpha| = \eta,\ \alpha_n < \eta'\}\\ \mathrm{Rep}_d(\eta,\eta') &= \{x^\alpha \mid |\alpha| = \eta, (\exists i \le n-1\ \alpha_i \ge d_i \text{ and } \alpha_n < \eta')\\ &\qquad \text{or } (\forall i \le n-1\ \alpha_i < d_i \text{ and } d_n \le \alpha_n < \eta' + d_n)\}.\end{aligned} \tag{15}$$

Next we define the subresultant matrix $\mathbf{J}_{\eta,\nu}(f)$.

Definition 3.1.2 *Let $f = (f_1, \ldots, f_n) \in \mathbf{R}[x_1, \ldots, x_n]$ be generic homogeneous polynomials of degrees $d = (d_1, \ldots, d_n)$. Fix η and ν such that $0 \le \delta - \nu \le \eta \le \delta - \eta \le \nu \le \delta$ and let $\eta' = \eta - (\delta - \nu)$. The $\mathbf{R}$-module homomorphism*

$$\mathbf{J}_{\eta,\nu}(f) : \langle \mathrm{Mon}(\eta,\eta')\rangle^* \oplus \langle \mathrm{Rep}_d(\delta-\eta)\rangle \to \langle \mathrm{Mon}(\delta-\eta)\rangle \oplus \langle \mathrm{Rep}_d(\eta,\eta')\rangle^*$$

corresponding to the subresultant matrix is defined as follows. Let $\Omega_{\eta,\eta'}$ be the restriction of Ω_η (defined in (6)) to $\langle \mathrm{Mon}(\eta,\eta')\rangle^*$. Let $\Phi^*_{\eta,\eta'}$ be the dual of the map

$\Phi_\eta|_{\langle \mathrm{Rep}_d(\eta,\eta')\rangle}$ *(defined in (7)) restricted to* $\langle \mathrm{Mon}(\eta,\eta')\rangle^*$*. Then* $\mathbf{J}_{\eta,\nu}(f)$ *is defined as*

$$\left(y^\alpha, x^\beta\right) \mapsto \left(\Omega_{\eta,\eta'}(y^\alpha) + \Phi_{\delta-\eta}(x^\beta),\ \ \Phi^*_{\eta,\eta'}(y^\alpha)\right)$$

for $y^\alpha \in \mathrm{Mon}(\eta,\eta')^*$ *and* $x^\beta \in \mathrm{Rep}_d(\delta-\eta)$*. Abusing the notation, we denote the matrix of the map* $\mathbf{J}_{\eta,\nu}(f)$ *again by* $\mathbf{J}_{\eta,\nu}(f)$*.*

Permuting rows and columns, the matrix $\mathbf{J}_{\eta,\nu}(f)$ has the following structure:

$$\mathbf{J}_{\eta,\nu}(f) = \begin{array}{cc} \begin{array}{cc} \mathrm{Mon}(\delta-\eta) & \mathrm{Rep}_d(\eta,\eta')^* \end{array} & \\ \left[\begin{array}{c|c} \Omega_{\eta,\eta'} & \Phi^*_{\eta,\eta'} \\ \hline \Phi_{\delta-\eta} & 0 \end{array}\right] & \begin{array}{l} \mathrm{Mon}(\eta,\eta')^* \\ \mathrm{Rep}_d(\delta-\eta) \end{array} \end{array}$$

As we mentioned earlier, the matrix $\mathbf{J}_{\eta,\nu}(f)$ is a submatrix of $\mathbf{J}_\eta(f)$, obtained by erasing the rows corresponding to the monomials in $\overline{\mathrm{Mon}}(\eta,\eta')$ and the columns corresponding to the monomials in $\overline{\mathrm{Rep}}_d(\eta,\eta')$. Therefore, the difference between the number of columns and rows of $\mathbf{J}_{\eta,\nu}(f)$ is

$$\begin{aligned} \#\overline{\mathrm{Mon}}(\eta,\eta') - \#\overline{\mathrm{Rep}}_d(\eta,\eta') &= \#\mathrm{Mon}(\eta-\eta') - \#\mathrm{Rep}_d(\eta-\eta') \\ &= \#\mathrm{Mon}(\delta-\nu) - \#\mathrm{Rep}_d(\delta-\nu) \\ &= \mathcal{H}_d(\delta-\nu) \\ &= \mathcal{H}_d(\nu). \end{aligned}$$

Example 2.1.1 (cont)
Let $n = 3$, $d = (3,3,2)$ and $f = (f_1, f_2, f_3)$ as in Example 2.1.1. As in the previous example we set $\eta = 2$. For $\nu = \delta = 5$ we have $\eta' = \eta - (\delta - \nu) = 2$, therefore we erase all rows of $\mathbf{J}_2(f)$ in (13) corresponding to monomials which have degree 2 in the variable w. That is, we erase the single row corresponding to w^2. Since $\overline{\mathrm{Rep}}_d(2,2) = \emptyset$,

we do not erase any columns. Thus the subresultant matrix $\mathbf{J}_{2,5}(f)$ has size 10×11:

$$\begin{bmatrix}
\mu_{u^2,x^3} & \mu_{u^2,x^2y} & \mu_{u^2,x^2z} & \mu_{u^2,xy^2} & \mu_{u^2,xyz} & \mu_{u^2,xz^2} & \mu_{u^2,y^3} & \mu_{u^2,y^2z} & \mu_{u^2,yz^2} & \mu_{u^2,z^3} & c_0 \\
\mu_{vu,x^3} & \mu_{vu,x^2y} & \mu_{vu,x^2z} & \mu_{vu,xy^2} & \mu_{vu,xyz} & \mu_{vu,xz^2} & \mu_{vu,y^3} & \mu_{vu,y^2z} & \mu_{vu,yz^2} & \mu_{vu,z^3} & c_1 \\
\mu_{wu,x^3} & \mu_{wu,x^2y} & \mu_{wu,x^2z} & \mu_{wu,xy^2} & \mu_{wu,xyz} & \mu_{wu,xz^2} & \mu_{wu,y^3} & \mu_{wu,y^2z} & \mu_{wu,yz^2} & \mu_{wu,z^3} & c_2 \\
\mu_{v^2,x^3} & \mu_{v^2,x^2y} & \mu_{v^2,x^2z} & \mu_{v^2,xy^2} & \mu_{v^2,xyz} & \mu_{v^2,xz^2} & \mu_{v^2,y^3} & \mu_{v^2,y^2z} & \mu_{v^2,yz^2} & \mu_{v^2,z^3} & c_3 \\
\mu_{wv,x^3} & \mu_{wv,x^2y} & \mu_{wv,x^2z} & \mu_{wv,xy^2} & \mu_{wv,xyz} & \mu_{wv,xz^2} & \mu_{wv,y^3} & \mu_{wv,y^2z} & \mu_{wv,yz^2} & \mu_{wv,z^3} & c_4 \\
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & 0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & 0 & 0 & 0 & 0 & 0 \\
b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & 0 \\
0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0 & 0 \\
0 & 0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0
\end{bmatrix}$$

For $\nu = 4$ we have $\eta' = 1$, therefore we erase all rows which correspond to monomials of degree at least 1 in the variable w. Again, $\overline{\mathrm{Rep}}_d(2,1) = \emptyset$, so we do not erase any columns. Thus the subresultant matrix $\mathbf{J}_{2,4}(f)$ has size 8×11:

$$\begin{bmatrix}
\mu_{u^2,x^3} & \mu_{u^2,x^2y} & \mu_{u^2,x^2z} & \mu_{u^2,xy^2} & \mu_{u^2,xyz} & \mu_{u^2,xz^2} & \mu_{u^2,y^3} & \mu_{u^2,y^2z} & \mu_{u^2,yz^2} & \mu_{u^2,z^3} & c_0 \\
\mu_{vu,x^3} & \mu_{vu,x^2y} & \mu_{vu,x^2z} & \mu_{vu,xy^2} & \mu_{vu,xyz} & \mu_{vu,xz^2} & \mu_{vu,y^3} & \mu_{vu,y^2z} & \mu_{vu,yz^2} & \mu_{vu,z^3} & c_1 \\
\mu_{v^2,x^3} & \mu_{v^2,x^2y} & \mu_{v^2,x^2z} & \mu_{v^2,xy^2} & \mu_{v^2,xyz} & \mu_{v^2,xz^2} & \mu_{v^2,y^3} & \mu_{v^2,y^2z} & \mu_{v^2,yz^2} & \mu_{v^2,z^3} & c_3 \\
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & 0 \\
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & 0 & 0 & 0 & 0 & 0 \\
b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & 0 \\
0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0 & 0 \\
0 & 0 & c_0 & 0 & c_1 & c_2 & 0 & c_3 & c_4 & c_5 & 0
\end{bmatrix} \tag{16}$$

and the rows correspond to the monomials

$$\begin{bmatrix} u^2 & vu & v^2 & x^3 & z^2x & y^3 & z^2y & z^3 \end{bmatrix},$$

while the columns still correspond to the monomials

$$\begin{bmatrix} x^3 & yx^2 & zx^2 & y^2x & zyx & z^2x & y^3 & y^2z & z^2y & z^3 & w^2 \end{bmatrix} \quad \boxdot$$

We will use the following lemma throughout the paper.

Lemma 3.1.3 *If Φ_η is the map defined in (7) then the restriction of Φ_η to $\langle \overline{\mathrm{Rep}}_d(\eta, \eta') \rangle$ has its image in $\langle \overline{\mathrm{Mon}}(\eta, \eta') \rangle$. In other words, the matrix Φ_η^* of the dual map has* *the following structure:*

$$\Phi^*_\eta = \begin{array}{cc} \begin{array}{cc} \mathrm{Rep}_d(\eta,\eta')^* & \overline{\mathrm{Rep}}_d(\eta,\eta')^* \end{array} & \\ \left[\begin{array}{c|c} \Phi^*_{\eta,\eta'} & \mathit{0} \\ \hline & \end{array}\right] & \begin{array}{l} \mathrm{Mon}(\eta,\eta')^* \\ \overline{\mathrm{Mon}}(\eta,\eta')^* \end{array} \end{array} .$$

Proof.
Let $x^\alpha \in \overline{\mathrm{Rep}}_d(\eta,\eta')$, i.e. $|\alpha| = \eta, \alpha_n \geq \eta'$ and either there exists $i \leq n-1$ such that $\alpha_i \geq d_i$ or $\alpha_n \geq \eta' + d_n$.
Case 1: The smallest index i such that $\alpha_i \geq d_i$ is not n. Since $x^\alpha \in \overline{\mathrm{Rep}}_d(\eta,\eta')$, the image

$$\Phi_\eta(x^\alpha) = \frac{x^\alpha}{x_i^{d_i}} \cdot f_i$$

has degree at least η' in x_n, therefore all terms of $\Phi_\eta(x^\alpha)$ are in $\overline{\mathrm{Mon}}(\eta,\eta')$.
Case 2: The smallest index i such that $\alpha_i \geq d_i$ is n. In this case $\alpha_n \geq d_n + \eta'$, thus

$$\deg_{x_n} \frac{x^\alpha}{x_n^{d_n}} \cdot f_n \geq \eta'.$$

Again, all terms of $\Phi_\eta(x^\alpha)$ are in $\overline{\mathrm{Mon}}(\eta,\eta')$. ∎

Lemma 3.1.4 *The matrix $\Phi^*_{\eta,\eta'}$ has at least as many rows as columns.*

Proof.
First consider the case when $d_n \leq \eta'$. Let

$$\begin{aligned} A &:= \mathrm{Mon}(\eta,\eta') - \mathrm{Rep}_d(\eta,\eta') = \{x^\alpha \mid |\alpha| = \eta,\ \forall i \leq n-1\ \ \alpha_i < d_i,\ \text{and } 0 \leq \alpha_n < d_n\} \\ B &:= \mathrm{Rep}_d(\eta,\eta') - \mathrm{Mon}(\eta,\eta') = \{x^\alpha \mid |\alpha| = \eta,\ \forall i \leq n-1\ \ \alpha_i < d_i,\ \text{and } \eta' \leq \alpha_n < \eta' + d_n\}. \end{aligned}$$

Note that $|A| = \mathcal{H}_d(\eta)$, $|B| = \mathcal{H}_d(\eta - \eta') = \mathcal{H}_d(\delta - \nu)$, and their difference is the difference between the number of rows and columns of $\Phi^*_{\eta,\eta'}$. By the assumption $0 \leq \delta - \nu \leq \eta \leq \delta - \eta \leq \nu \leq \delta$ we have $\mathcal{H}_d(\delta - \nu) \leq \mathcal{H}_d(\eta)$ (using the fact that

$\mathcal{H}_d(t)$ is monotonically increasing in the interval $[0, \lfloor \frac{\delta}{2} \rfloor])$.
In the case when $\eta' < d_n$ we have

$$\begin{aligned} A &= \mathrm{Mon}(\eta, \eta') - \mathrm{Rep}_d(\eta, \eta') = \{x^\alpha \mid |\alpha| = \eta,\ \forall i \le n-1\ \ \alpha_i < d_i,\ \text{and } 0 \le \alpha_n < \eta'\} \\ B &= \mathrm{Rep}_d(\eta, \eta') - \mathrm{Mon}(\eta, \eta') = \{x^\alpha \mid |\alpha| = \eta,\ \forall i \le n-1\ \ \alpha_i < d_i,\ \text{and } d_n \le \alpha_n < \eta' + d_n\}. \end{aligned}$$

In this case $|A| = \mathcal{H}_{d'}(\eta)$ and $|B| = \mathcal{H}_{d'}(\eta - d_n)$ where $d' = (d_1, \ldots, d_{n-1}, \eta')$. Let $\delta' := \sum_{i=1}^{n-1}(d_i - 1) + (\eta' - 1)$. Then it is easy to check that $\eta \le \nu$ implies that either $\eta \le \lfloor \frac{\delta'}{2} \rfloor$ or $\eta - d_n \le \delta' - \eta \le \lfloor \frac{\delta'}{2} \rfloor$. This implies that $\mathcal{H}_{d'}(\eta - d_n) \le \mathcal{H}_{d'}(\eta)$ (using the fact that $\mathcal{H}_{d'}(t) = \mathcal{H}_{d'}(\delta' - t)$ and the monotonicity of $\mathcal{H}_{d'}(t)$ in $[0, \lfloor \frac{\delta'}{2} \rfloor])$. ■

Definition 3.1.5 *Let $T \subseteq \mathrm{Mon}(\delta - \eta)$ of cardinality $\mathcal{H}_d(\nu)$. Denote by $\mathbf{M}_T^{\eta,\nu}(f)$ the maximal square submatrix of $\mathbf{J}_{\eta,\nu}(f)$ with columns not corresponding to monomials in T.*

Example 2.1.1 (cont)
Continuing the previous example, for different T's the matrix $\mathbf{M}_T^{2,4}(f)$ can be any maximal square submatrix of $\mathbf{J}_{2,4}(f)$ in (16) which contains the last column. In this case $\mathcal{H}_{(3,3,2)}(4) = 3$, therefore $T \subset \mathrm{Mon}(3)$ must have cardinality 3. For example $T := \{x^3, y^3, z^3\}$ we get that $\mathbf{M}^{2,4}_{\{x^3,y^3,z^3\}}(f)$ is the following 8×8 matrix

$$\begin{bmatrix} \mu_{u^2,x^2y} & \mu_{u^2,x^2z} & \mu_{u^2,xy^2} & \mu_{u^2,xyz} & \mu_{u^2,xz^2} & \mu_{u^2,y^2z} & \mu_{u^2,yz^2} & c_0 \\ \mu_{vu,x^2y} & \mu_{vu,x^2z} & \mu_{vu,xy^2} & \mu_{vu,xyz} & \mu_{vu,xz^2} & \mu_{vu,y^2z} & \mu_{vu,yz^2} & c_1 \\ \mu_{v^2,x^2y} & \mu_{v^2,x^2z} & \mu_{v^2,xy^2} & \mu_{v^2,xyz} & \mu_{v^2,xz^2} & \mu_{v^2,y^2z} & \mu_{v^2,yz^2} & c_3 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_7 & a_8 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_7 & b_8 & 0 \\ c_0 & 0 & c_1 & c_2 & 0 & c_4 & c_5 & 0 \\ 0 & c_0 & 0 & c_1 & c_2 & c_3 & c_4 & 0 \end{bmatrix} \quad \boxdot$$

In the following proposition we prove that there exists T such that $\mathbf{M}_T^{\eta,\nu}(f)$ is generically non-singular.

Proposition 3.1.6 *Let $n \ge 2$, $f = (f_1, \ldots, f_n)$ be generic and let δ, ν, η and $\eta' = \eta - (\delta - \nu)$ be as above, and assume that $0 \le \delta - \nu \le \eta \le \delta - \eta \le \nu \le \delta$. Then there exists $T \subseteq \mathrm{Mon}(\delta - \eta)$ of cardinality $\mathcal{H}_d(\nu)$ such that for generic f the matrix $\mathbf{M}_T^{\eta,\nu}(f)$ is non-singular.*

Proof.
For a fixed T, to prove that $\mathbf{M}_T^{\eta,\nu}(f)$ is non-singular for generic f, it is sufficient to find a specific system $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ of degree $(d_1, \ldots, d_n)$ such that

$\det(\mathbf{M}_T^{\eta,\nu}(\tilde{f})) \neq 0$.

First consider the system $\tilde{f} := (x_1^{d_1}, \ldots, x_n^{d_n})$. Then Jouanolou's matrix $\mathbf{J}_\eta(\tilde{f})$ corresponds to the identity map (cf. [12]). The matrix $\mathbf{M}_T^{\eta,\nu}(\tilde{f})$ is obtained from $\mathbf{J}_\eta(\tilde{f})$ by deleting rows corresponding to $\overline{\mathrm{Mon}}(\eta, \eta')$ and columns corresponding to $T \cup \overline{\mathrm{Rep}}_d(\eta, \eta')$. Unfortunately, the removal of the rows $\overline{\mathrm{Mon}}(\eta, \eta')$ may leave the submatrix $\Phi^*_{\eta,\eta'}(\tilde{f})$ of deficient rank with zero columns.

We shall construct a system $\tilde{f}' := (x_1^{d_1}, \ldots, x_{n-1}^{d_{n-1}}, \tilde{p})$ for some $\tilde{p} \in k[x_1, \ldots, x_n]_{d_n}$ such that $\Phi^*_{\eta,\eta'}(\tilde{f}')$ has full rank. Let

$$C := \{x^\alpha \mid |\alpha| = \eta,\ \forall i \leq n-1\ \ \alpha_i < d_i,\ \text{and } d_n \leq \alpha_n < \eta' + d_n\}$$
$$R := \{x^\alpha \mid |\alpha| = \eta,\ \forall i \leq n-1\ \ \alpha_i < d_i,\ \text{and } 0 \leq \alpha_n < \eta'\}.$$

Note that $\frac{1}{x_n^{d_n}} \cdot C = H_{d'}(\eta - d_n)$ and $R = H_{d'}(\eta)$ where $d' = (d_1, \ldots d_{n-1}, \eta')$. Also note that C is the set of monomials multiplied by $f_n / x_n^{d_n}$ in the map $\Phi_{\eta,\eta'}$ and that C contains the set of monomials corresponding to the zero columns in $\Phi^*_{\eta,\eta'}(\tilde{f})$. Consider the $\mathbf{R}$-module homomorphism

$$\begin{aligned} \psi_p : \langle H_{d'}(\eta - d_n)\rangle &\to \langle H_{d'}(\eta)\rangle \\ x^\alpha &\mapsto x^\alpha \cdot p \qquad \bmod \langle x_1^{d_1}, \ldots, x_n^{\eta'}\rangle_\eta \end{aligned}$$

By [15, Corollary 3.5 and Theorem 3.8.(0)], if we take

$$\tilde{p} := (x_1 + \cdots + x_n)^{d_n}$$

then the matrix of the map $\psi_{\tilde{p}}$ has full rank. For $\delta' := \delta - d_n + \eta'$, the inequality $\eta \leq \nu$ implies that either $\eta \leq \lfloor \frac{\delta'}{2} \rfloor$ or $\eta - d_n \leq \delta' - \eta \leq \lfloor \frac{\delta'}{2} \rfloor$. Using the fact that $\mathcal{H}_{d'}(t) = \mathcal{H}_{d'}(\delta' - t)$ and that $\mathcal{H}_{d'}(t)$ is monotonically increasing in $[0, \lfloor \frac{\delta'}{2} \rfloor])$ we get that $\mathcal{H}_{d'}(\eta - d_n) \leq \mathcal{H}_{d'}(\eta)$. Therefore, the map $\psi_{\tilde{p}}$ is injective.

For $\tilde{f}' := (x_1^{d_1}, \ldots, x_{n-1}^{d_{n-1}}, \tilde{p})$ the matrix $\Phi^*_{\eta,\eta'}(\tilde{f}')$ has a block triangular form with a block of the identity matrix corresponding to the columns $\mathrm{Rep}_d(\eta, \eta') - C$ and a block of the map $\psi_{\tilde{p}}$ corresponding to the columns C. Therefore $\Phi^*_{\eta,\eta'}(\tilde{f}')$ has full column rank.

Finally, the matrix $\mathbf{J}_\eta(\tilde{f}')$ has full row rank (note that the Bezoutian of $\tilde{f}'$ is the same as the Bezoutian of $\tilde{f} = (x_1^{d_1}, \ldots, x_n^{d_n})$). This implies that $\mathbf{J}_{\eta,\nu}(\tilde{f}')$ has also full row rank (using Lemma 3.1.3). Since we just proved that the columns of $\mathbf{J}_{\eta,\nu}(\tilde{f}')$ corresponding to $\mathrm{Rep}_d(\eta, \eta')$ are linearly independent, therefore there exists a subset T of $\mathrm{Mon}(\delta-\eta)$ of cardinality $\mathcal{H}_d(\nu)$ such that after erasing the columns of $\mathbf{J}_{\eta,\nu}(\tilde{f}')$ corresponding to T we get a nonsingular matrix $\mathbf{M}_T^{\eta;\nu}(\tilde{f}')$. ■

3.2 Definition of subresultants

In this subsection we define square submatrices of Jouanolou's resultant matrix $\mathbf{J}_\eta(f)$ (see Definition 2.2.5) such that the ratio of their determinants gives the subresultant.

As in Definition 3.1.5, fix $T \subseteq \mathrm{Mon}(\delta-\eta)$ of cardinality $\mathcal{H}_d(\nu)$ and denote by $\mathbf{M}_T^{\eta;\nu}$ the submatrix of $\mathbf{J}_{\eta,\nu}(f)$ with columns not belonging to T. Similarly as in Theorem 2.2.6, for $t \geq 0$ let $\mathbf{E}_t$ denote the submatrix of Φ_t (see Definition 2.2.5) with rows and columns corresponding to monomials in $\mathrm{Dod}_d(t)$ (see Definition 2.2.2). We define $\mathbf{E}_{\eta,\eta'}$ to be the submatrix of Φ^*_η such that its rows and columns correspond to $\mathrm{Dod}_d(\eta) \cap \mathrm{Rep}_d(\eta,\eta')$. Note that $\mathbf{E}_{\eta,\eta'}$ is a submatrix of $\Phi^*_{\eta,\eta'}$, since for $x^\alpha \in \mathrm{Dod}_d(\eta) \cap \mathrm{Rep}_d(\eta,\eta')$ there exists $i < n$ such that $\alpha_i \geq d_i$ therefore $\alpha_n < \eta'$ by the definition of $\mathrm{Rep}_d(\eta,\eta')$ (see (15)), thus $x^\alpha \in \mathrm{Mon}(\eta,\eta')$. Also, by Lemma 3.1.3 we have that

$$\det(\mathbf{E}_{\eta,\eta'}) = \frac{\det(\mathbf{E}_\eta)}{\det(\mathbf{E}_{\delta-\nu})}.$$

Moreover, both $\mathbf{E}_{\delta-\eta}$ and $\mathbf{E}_{\eta,\eta'}$ are generically non-singular (cf. [13]).

Definition 3.2.1 *Using the above definitions of $\mathbf{M}_T^{\eta;\nu}$, $\mathbf{E}_{\delta-\eta}$ and $\mathbf{E}_{\eta,\eta'}$ we define the subresultant $\Gamma_T^{\eta;\nu}(f)$ corresponding to T by*

$$\Gamma_T^{\eta,\nu}(f) := \frac{\det(\mathbf{M}_T^{\eta;\nu})}{\det(\mathbf{E}_{\delta-\eta})\det(\mathbf{E}_{\eta,\eta'})}. \tag{17}$$

Example 2.1.1 (cont)
Continuing the previous example, we have $\mathrm{Dod}_d(t) = \emptyset$ for any $t \leq 5$, therefore, if $0 < \eta < 5$, then the denominator of (17) is 1. For $\eta = 0$, Jouanolou's matrix contains a single row of Bezoutian type, therefore there is only one possible subresultant matrix $\mathbf{J}_{0,5}$ obtained by removing this one row. Then $\mathbf{J}_{0,5}$ is a Macaulay type subresultant matrix, which has size 20×21. Note that for $\nu = \delta - \eta$ we always get a Macaulay type subresultant matrix. We cannot include here $\mathbf{J}_{0,5}$, only $\mathbf{E}_5$. Since $\mathrm{Dod}_{(3,3,2)}(5) = \{x^3z^2, y^3z^2\}$, therefore $\mathbf{E}_5$ has size 2×2:

$$\begin{bmatrix} a_0 & a_6 \\ b_0 & b_6 \end{bmatrix}$$

Thus, for any $T \subset \mathrm{Mon}(5)$, $|T| = 1$, we have

$$\Gamma_T^{0,5}(f) = \frac{\det(\mathbf{M}_T^{0,5})}{a_0b_6 - a_6b_0} \quad \boxdot$$

First we show that $\Gamma_T^{\eta,\nu}$ is a polynomial in the coefficients of $f_1, \ldots, f_n$.

Proposition 3.2.2 *Let $f_i = \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha$ be polynomials with parametric coefficients for $1 \leq i \leq n$, and let δ, ν, η, η', T be as above. Assume that $0 \leq \delta - \nu \leq \eta \leq \delta - \eta \leq \nu \leq \delta$. Then $\Gamma_T^{\eta,\nu}(f)$ is a polynomial in the coefficients $c_{i,\alpha}$ $(|\alpha| = d_i)$.*

Proof. Similarly as in [6, Lemma 3.4], using the block structure of the matrix $\mathbf{M}_T^{\eta,\nu}$, we can write

$$\det(\mathbf{M}_T^{\eta,\nu}) = \sum_{S_1,S_2} \epsilon_{S_1,S_2} \cdot m_{S_1} \cdot m_{S_2} \cdot m^c_{S_1,S_2} \tag{18}$$

where the summation runs through all subsets $S_1 \subset \mathrm{Mon}(\delta - \eta) - T$ and $S_2 \subset \mathrm{Mon}(\eta, \eta')$ both of cardinality $\mathcal{H}_d(\eta) - \mathcal{H}_d(\nu)$. Here $\epsilon_{S_1,S_2} = \pm 1$, m_{S_1} is the determinant of the submatrix of $\Phi_T^{\delta-\eta}$ with columns not corresponding to S_1, m_{S_2} is the determinant of the submatrix of $\Phi^*_{\eta,\eta'}$ with rows not corresponding to S_2, and $m^c_{S_1,S_2}$ is the minor of $\Omega_T^{\eta,\eta'}$ with columns corresponding to S_1 and rows corresponding to S_2. Here $\Phi_T^{\delta-\eta}$ and $\Omega_T^{\eta,\eta'}$ denotes the submatrices of $\Phi_{\delta-\eta}$ and $\Omega_{\eta,\eta'}$ respectively, such that the columns corresponding to T are removed. Note that $\mathcal{H}_d(\eta) - \mathcal{H}_d(\nu) \geq 0$ by the assumption $0 \leq \eta \leq \delta - \eta \leq \nu \leq \delta$ (cf. [6]).

To prove that $\Gamma_T^{\eta,\nu}(f)$ is a polynomial, first note that for all $S_1 \subset \mathrm{Mon}(\delta - \eta) - T$ of cardinality $\mathcal{H}_d(\eta) - \mathcal{H}_d(\nu)$ we have

$$m_{S_1} = \det(\mathbf{E}_{\delta-\eta}) \cdot \Delta^{\delta-\eta}_{S_1 \cup T}$$

where $\Delta^{\delta-\eta}_{S_1\cup T}$ is a Macaulay type subresultant and is a polynomial by [3]. Therefore $\det(\mathbf{E}_{\delta-\eta})$ divides m_{S_1} for all S_1 in the summation in (18).

On the other hand, to prove that $\det(\mathbf{E}_{\eta,\eta'})$ divides m_{S_2}, note that by Lemma 3.1.3 the matrix Φ^*_η has a block-triangular structure. Therefore, for every $S_2 \subset \mathrm{Mon}(\eta, \eta')$ of cardinality $\mathcal{H}_d(\eta) - \mathcal{H}_d(\nu)$ and every $S_3 \subset \overline{\mathrm{Mon}}(\eta, \eta')$ of cardinality $\mathcal{H}_d(\nu)$, the determinant of the submatrix of Φ^*_η with rows not corresponding to $S_2 \cup S_3$ is

$$m_{S_2 \cup S_3} = m_{S_2} \cdot m_{S_3}.$$

where m_{S_3} is the minor of $\Phi^*_{\delta-\nu}$ with rows not corresponding to S_3. But $m_{S_3 \cup S_2} = \det(\mathbf{E}_\eta) \cdot \Delta^\eta_{S_2 \cup S_3}$ and $m_{S_3} = \det(\mathbf{E}_{\delta-\nu}) \cdot \Delta^{\delta-\nu}_{S_3}$, therefore

$$m_{S_2} = \frac{\det(\mathbf{E}_\eta)\Delta^\eta_{S_2\cup S_3}}{\det(\mathbf{E}_{\delta-\nu})\Delta^{\delta-\nu}_{S_3}} = \det(\mathbf{E}_{\eta,\eta'}) \frac{\Delta^\eta_{S_2\cup S_3}}{\Delta^{\delta-\nu}_{S_3}}. \tag{19}$$

Now we apply the same trick as in [6, Theorem 3.2]. We can use two different sets of parameters $(b_{i,\alpha})|_{|\alpha|=d_i}$ and $(c_{i,\alpha})|_{|\alpha|=d_i}$ to define two generic polynomial systems f^b and f^c and to consider the matrix

$$
\Phi^*_\eta(f^b, f^c) = \begin{array}{cc|c}
\mathrm{Rep}(\eta,\eta') & \overline{\mathrm{Rep}}(\eta,\eta') & \\
\left[\begin{array}{c|c}
\Phi^*_{\eta,\eta'}(f^b) & 0 \\
\hline
 & \Phi^*_{\delta-\nu}(f^c)
\end{array}\right. & \left.\right] & \begin{array}{c}\mathrm{Mon}(\eta,\eta') \\ \overline{\mathrm{Mon}}(\eta,\eta')\end{array}
\end{array}.
$$

Now

$$
m_{S_2}(f^b) = \det(\mathbf{E}_{\eta,\eta'}(f^b)) \cdot \frac{\Delta^\eta_{S_2\cup S_3}(f^b, f^c)}{\Delta^{\delta-\nu}_{S_3}(f^c)}
$$

and both sides are polynomials in $(b_{i,\alpha})$ and $(c_{i,\alpha})$, so we deduce that $\Delta^{\delta-\nu}_{S_3}(f^c)$ divides $\Delta^\eta_{S_2\cup S_3}(f^b, f^c)$, therefore $\det(\mathbf{E}_{\eta,\eta'})(f^b)$ divides $m_{S_2}(f^b)$. This proves that $\Gamma^{\eta;\nu}_T(f)$ is a polynomial. $\blacksquare$

Next we prove that $\Gamma^{\eta;\nu}_T(f)$ has the same degree in the coefficients of f_i $(1 \leq i \leq n)$ as the Macaulay type subresultants Δ^ν_S $(S \subseteq \mathrm{Mon}(\nu),\ |S| = \mathcal{H}_d(\nu))$.

Proposition 3.2.3 *Let* $f_i = \sum_{|\alpha|=d_i} c_{i,\alpha}x^\alpha$ *for* $1 \leq i \leq n$*, and let* δ*,* ν*,* η*,* η'*,* T *and* $\Gamma^{\eta;\nu}_T(f)$ *be as above. Assume that* $0 \leq \eta \leq \delta - \eta \leq \nu \leq \delta$*. Then for any fixed* $1 \leq i \leq n$*,* $\Gamma^{\eta;\nu}_T(f)$ *is homogeneous in the coefficients* $c_{i,\alpha}$ $(|\alpha| = d_i)$ *of degree* $\mathcal{H}_{\hat{d}^i}(\nu - d_i)$*. As before,* $\mathcal{H}_{\hat{d}^i}$ *denotes the Hilbert function of a regular sequence with* $n-1$ *homogeneous polynomials in* n *variables with degrees* $\hat{d}^i = (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n)$*.*

Proof.SIMPLIFY!!! As in the previous proof we write

$$
\det(\mathbf{M}^{\eta,\nu}_T) = \sum_{S_1,S_2} \epsilon_{S_1,S_2} \cdot m_{S_1} \cdot m_{S_2} \cdot m^c_{S_1,S_2}
$$

using the same notation as in (18).

We define the sets

$$J_i(t) := \{x^\alpha \mid |\alpha| = t,\ \alpha_i \geq d_i \quad \text{and} \quad \forall j \neq i\ \alpha_j < d_j\}.$$

for $1 \leq i \leq n$ and $t \geq 0$. We claim that

$$\begin{aligned}
\deg_{c_{i,\alpha}}(m_{S_1}) - \deg_{c_{i,\alpha}}(\det(\mathbf{E}_{\delta-\eta})) &= \#J_i(\delta - \eta) && (20)\\
\deg_{c_{i,\alpha}}(m_{S_2}) - \deg_{c_{i,\alpha}}(\det(\mathbf{E}_{\eta,\eta'})) &= \#J_i(\eta) - \#J_i(\eta - \eta') && (21)\\
\deg_{c_{i,\alpha}}(m^c_{S_1,S_2}) &= \mathcal{H}_d(\eta) - \mathcal{H}_d(\nu). && (22)
\end{aligned}$$

Equation (20) was proved in [3]. Equation (22) follows from the fact that each entry of $\Omega_{\eta,\eta'}$ has degree 1 in $c_{i,\alpha}$. To prove equation (21) we denote

$$\begin{aligned}
\mathrm{Rep}_d^{(i)}(\eta,\eta') &:= \{x^\alpha \mid |\alpha| = \eta,\ \ \alpha_i \geq d_i,\ \forall j < i\ \alpha_j < d_j,\ \ \alpha_n < \eta'\} \quad i \leq n-1\\
\mathrm{Rep}_d^{(n)}(\eta,\eta') &:= \{x^\alpha \mid |\alpha| = \eta,\ \forall j < n\ \alpha_j < d_j,\ d_n \leq \alpha_n < \eta' + d_n\}.
\end{aligned}$$

Then clearly

$$\deg_{c_{i,\alpha}}(m_{S_2}) - \deg_{c_{i,\alpha}}(\det(\mathbf{E}_{\eta,\eta'})) = \#\left(\mathrm{Rep}_d^{(i)}(\eta,\eta') - \mathrm{Dod}_d(\eta)\right) = \#J_i(\eta) - \#J_i(\eta - \eta')$$

for all $1 \leq i \leq n$, which proves (21).

Therefore,

$$\deg_{c_{i,\alpha}}(\Gamma_T^{\eta,\nu}(f)) = \#J_i(\eta) + \#J_i(\delta - \eta) + \mathcal{H}_d(\eta) - \#J_i(\delta - \nu) - \mathcal{H}_d(\delta - \nu).$$

Define the sets

$$\begin{aligned}
H_d(t) &:= \{x^\alpha \mid |\alpha| = t,\ \forall j\ \alpha_j < d_j\}\\
H_{\hat{d}^i}(t) &:= \{x^\alpha \mid |\alpha| = t,\ \forall j \neq i\ \alpha_j < d_j\}
\end{aligned}$$

of cardinalities $\mathcal{H}_d(t)$ and $\mathcal{H}_{\hat{d}^i}(t)$, respectively. Also, for $t' \leq t$ we define the set

$$H_{\hat{d}^i}(t,t') := \{x^\alpha \mid |\alpha| = t,\ \alpha_i < t',\ \ \forall j \neq i\ \alpha_j < d_j\}$$

of cardinality $\mathcal{H}_{\hat{d}^i}(t) - \mathcal{H}_{\hat{d}^i}(t - t')$.

First we consider the case when $\eta \leq \nu - d_i$. We give a bijection between

$$H_{\hat{d}^i}(\nu - d_i) \;\leftrightarrow\; H_{\hat{d}^i}(\eta, \eta') \cup^* J_i(\delta - \eta). \tag{23}$$

Let $x^\alpha \in H_{\hat{d}^i}(\nu - d_i)$. If $\sum_{j \neq i} \alpha_j \leq \delta - \eta - d_i$ then for

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \delta - \eta - \sum_{j \neq i} \alpha_j, \alpha_{i+1}, \ldots, \alpha_n)$$

$x^{\alpha'}$ is in $J_i(\delta - \eta)$ (since $\delta - \eta - \sum_{j \neq i} \alpha_j \geq d_i$). Moreover, since $\delta - \eta - d_i \leq \nu - d_i$, we get all the elements of $J_i(\delta - \eta)$ this way.
On the other hand, assume that $\delta - \eta - d_i < \sum_{j \neq i} \alpha_j \leq \nu - d_i$. Define $\alpha'_j := d_j - 1 - \alpha_j$ for all $j \neq i$. Then

$$\eta + 1 > \sum_{j \neq i} \alpha'_j \geq \delta - \nu + 1,$$

therefore by defining $\alpha'_i := \eta - \sum_{j \neq i} \alpha'_j$ we have that $\alpha'_i \leq \eta - (\delta - \nu + 1) < \eta'$, thus $x^{\alpha'} \in H_{\hat{d}^i}(\eta, \eta')$. Moreover, since $\eta \leq \nu - d_i$, all the elements of $H_{\hat{d}^i}(\eta, \eta')$ can be obtained this way, which gives the bijection in (23).

To obtain the claim of the proposition for the $\eta \leq \nu - d_i$ case, we assert that

$$\begin{aligned}
\mathcal{H}_{\hat{d}^i}(\nu - d_i) &= \# J_i(\delta - \eta) + \mathcal{H}_{\hat{d}^i}(\eta, \eta') \\
&= \# J_i(\delta - \eta) + \mathcal{H}_{\hat{d}^i}(\eta) - \mathcal{H}_{\hat{d}^i}(\delta - \nu) \\
&= \# J_i(\delta - \eta) + \# J_i(\eta) + \mathcal{H}_d(\eta) - \# J_i(\delta - \nu) - \mathcal{H}_d(\delta - \nu) \\
&= \deg_{c_{i,\alpha}}(\Gamma_T^{\eta,\nu}(f))
\end{aligned}$$

using the fact that $H_{\hat{d}^i}(t) \; = J_i(t) \cup H_d(t)$ for $t \geq 0$.

Secondly, we consider the case when $\eta > \nu - d_i$. We give a bijection between the discrete unions

$$H_{\hat{d}^i}(\nu - d_i) \cup^* H_{\hat{d}^i}(\delta - \nu) \;\leftrightarrow\; H_{\hat{d}^i}(\eta) \cup^* J_i(\delta - \eta). \tag{24}$$

Let $x^\alpha \in H_{\hat{d}^i}(\eta)$. If $\sum_{j \neq i} \alpha_j \leq \nu - d_i$ then for

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \nu - d_i - \sum_{j \neq i} \alpha_j, \alpha_{i+1}, \ldots, \alpha_n)$$

we have $x^{\alpha'} \in H_{\hat{d}^i}(\nu - d_i)$. Moreover, since $\nu - d_i < \eta$, we get a bijection between the sets

$$\{x^\alpha \in H_{\hat{d}^i}(\eta) \mid \sum_{j \neq i} \alpha_j \leq \nu - d_i\} \leftrightarrow H_{\hat{d}^i}(\nu - d_i). \tag{25}$$

On the other hand, assume that $\nu - d_i < \sum_{j\neq i}\alpha_j \leq \eta$. Define $\alpha'_j := d_j - 1 - \alpha_j$ for all $j \neq i$. Then

$$\delta - \nu + 1 > \sum_{j\neq i}\alpha'_j \geq \delta - \eta - d_i + 1.$$

Therefore, by defining $\alpha'_i := \delta - \nu - \sum_{j\neq i}\alpha'_j$ we have that $\alpha'_i < \eta - \nu + d_i$, thus

$$x^{\alpha'} \in H_{\hat{d}^i}(\delta-\nu) - \{x^\beta \mid |\beta| = \delta - \nu,\ \beta_i \geq \eta - \nu + d_i, \forall j \neq i\ \ \beta_j < d_j\}.$$

Observing that

$$\{x^\beta \mid |\beta| = \delta - \nu,\ \beta_i \geq \eta - \nu + d_i, \forall j \neq i\ \ \beta_j < d_j\} = H_{\hat{d}^i}(\delta - \eta - d_i)$$

and that

$$\frac{1}{x_i^{d_i}} \cdot J_i(\delta-\eta) = H_{\hat{d}^i}(\delta-\eta-d_i)$$

we get a bijection between

$$\{x^\alpha \in H_{\hat{d}^i}(\eta) \mid \nu - d_i < \sum_{j\neq i}\alpha_j\} \cup J_i(\delta-\eta) \leftrightarrow H_{\hat{d}^i}(\delta-\nu). \tag{26}$$

The bijections in (25) and in (26) give the bijection in (24). Again, we obtained that

$$\mathcal{H}_{\hat{d}^i}(\nu - d_i) = \#J_i(\delta-\eta) + \mathcal{H}_{\hat{d}^i}(\eta) - \mathcal{H}_{\hat{d}^i}(\delta-\nu) = \deg_{c_{i,\alpha}}(\Gamma_T^{\eta,\nu}(f)).$$

This proves the claim of the proposition in the $\eta > \nu - d_i$ case. ∎

The next proposition states that the non-vanishing of $\Gamma_T^{\eta,\nu}(f)$ implies that certain polynomials with $\mathcal{H}_d(\nu)+1$ terms are in the ideal $\langle f_1, \ldots, f_n\rangle$. This property allows the subresultants to be used in the solution of polynomial systems (see [10] and [16]).

Proposition 3.2.4 *Let $f_1, \ldots, f_n \subset \mathbf{R}[x_1, \ldots, x_n]$ be generic polynomials, and let δ, ν, η and $\eta' = \eta - (\delta - \nu)$ be as above. For any fixed $T \subset \mathrm{Mon}(\delta-\eta)$ of cardinality $\mathcal{H}_d(\nu)$ the following statements hold:*

(1) for all $x^\alpha \in \mathrm{Mon}(\delta-\eta) - T$

$$x_n^{\eta'}\left(\Gamma_T^{\eta,\nu}x^\alpha + \sum_{x^\beta\in T}\epsilon_\beta \Gamma^{\nu,\eta}_{(T\cup\{x^\alpha\}-\{x^\beta\})}x^\beta\right) \in \langle f_1(x), \ldots, f_n(x)\rangle_\nu \tag{27}$$

where $\epsilon_\beta = \pm 1$ and $\langle f_1(x), \ldots, f_n(x)\rangle_\nu$ denotes the degree ν homogeneous part of the ideal $\langle f_1(x), \ldots, f_n(x)\rangle$.

(2) For all $x^\alpha \in \mathrm{Mon}(\delta-\eta) - T$ *and for all* $x^\gamma \in \mathrm{Mon}(\eta+1)$

$$x^\gamma \left(\Gamma_T^{\eta,\nu} x^\alpha + \sum_{x^\beta \in T} \epsilon_\beta \Gamma^{\nu,\eta}_{(T\cup\{x^\alpha\}-\{x^\beta\})} x^\beta \right) \in \langle f_1(x),\ldots,f_n(x)\rangle_{\delta+1} \tag{28}$$

where $\epsilon_\beta = \pm 1$.

To prove Proposition 3.2.4 we need two lemmas.

Lemma 3.2.5 *Let* $\mathbf{R}$ *be a domain with fraction field* $\mathbf{K}$ *and let* $\mathcal{I} \subset \mathbf{K}[x_1,\ldots,x_n]$ *be an ideal. Let*

$$M = \left[\begin{array}{c|c} \quad B \quad & C \end{array}\right]$$

be a matrix, where $B = (b_{i,j}) \in \mathbf{R}^{l\times m}$ *and* $C = (c_{i,m+j}) \in \mathbf{R}^{l\times s}$. *Suppose that the columns of* B *correspond to the monomials* $(x^{\alpha(1)} \ldots, x^{\alpha(m)})$. *Assume that there exist elements* $(a_1,\ldots,a_s)$ *of* $\mathbf{K}$ *such that for all* $1 \le i \le l$ *we have*

$$\sum_{j=1}^{m} b_{i,j} x^{\alpha(j)} + \sum_{j=1}^{s} a_j c_{i,m+j} \quad \in \mathcal{I}. \tag{29}$$

Fix r *rows of* M *for some* $s+1 \le r \le \min(l, m+s)$. *Then for any* $S \subset \{1,\ldots,m+s\}$, *such that* $|S| = r-1$ *and* $\{m+1,\ldots,m+s\} \subset S$, *we have*

$$\sum_{j\notin S} (-1)^{\sigma(j,S)}\, \mathcal{D}_{S\cup\{j\}}\, x^{\alpha(j)} \;\in\; \mathcal{I}, \tag{30}$$

where $\mathcal{D}_X$ *denotes the determinant of the submatrix of* M *corresponding to the fixed* r *rows and the columns indexed by* X *for any set* $X \subset \{1,\ldots,m+s\}$ *with cardinality* $|X| = r$, *and* $\sigma(j,S)$ *denotes the ordinal number of* j *in the ordered set* $S \cup \{j\}$.

Remark 3.2.6 Note that the polynomials $\sum_{j\notin S} \pm\, \mathcal{D}_{S\cup\{j\}}\, x^{\alpha(j)}$ in (30) do not depend on the elements $(a_1,\ldots,a_s)$. In order the claim to be true it is sufficient that such elements exist.

Proof of Lemma 3.2.5.
We can assume without loss of generality that M consists of only r rows. Note that condition (29) is equivalent to the fact that for any $I = \{i_1,\ldots,i_{s+1}\} \subset \{1,\ldots,r\}$ we have

$$\psi_I(x) := \det\left|\begin{array}{c|ccc} \mathbf{b}_{i_1}(x) & \ldots & \mathbf{c}_{i_1} & \ldots \\ \vdots & & & \\ \mathbf{b}_{i_{s+1}}(x) & \ldots & \mathbf{c}_{i_{s+1}} & \ldots \end{array}\right| \in \mathcal{I},$$

where $\mathbf{b}_i(x) := \sum_{j=1}^m b_{i,j} x^{\alpha(j)}$ and $\mathbf{c}_i$ is the i-th row of C.

Fix any subset S of cardinality $r-1$ such that $\{m+1,\ldots,m+s\} \subset S \subset \{1,\ldots,m+s\}$ and denote by $S' := S \cap \{1,\ldots,m\}$. Then the claim of the lemma follows from

$$\sum_{j\notin S}(-1)^{\sigma(j,S)}\,\mathcal{D}_{S\cup\{j\}}\;x^{\alpha(j)} = \sum_{\substack{I\subset\{1,\ldots,r\}\\|I|=s+1}}(-1)^{\sigma(I)}\det(B_{\bar{I},S'})\psi_I(x), \tag{31}$$

where for each subset $I = \{i_1,\ldots,i_{s+1}\} \subset \{1,\ldots,r\}$, $B_{\bar{I},S'}$ denotes the submatrix of B with rows indexed by $\bar{I} := \{1,\ldots,r\} - I$ and with columns indexed by S'. (31) can be proved by using a straightforward linear algebra argument. ■

Lemma 3.2.7 *Let $f_1,\ldots,f_n$, δ, ν, η, η' be as above and consider the map $\Phi^*_{\eta,\eta'}$: $\langle \mathrm{Mon}(\eta,\eta')\rangle^* \to \langle \mathrm{Rep}_d(\eta,\eta')\rangle^*$ defined in Definition 3.1.2. Denote by D the column vector*

$$\left(x^\gamma \mathrm{Morl}_\beta(x)\right)_{y^\beta\in\mathrm{Mon}^*(\eta,\eta')},$$

where x^γ is any fixed element of $\{x_n^{\eta'}\} \cup \mathrm{Mon}(\eta+1)$. Then any maximal minor of the $(\#\mathrm{Mon}(\eta,\eta')) \times (\#\mathrm{Rep}_d(\eta,\eta')+1)$ matrix

$$\boxed{\begin{array}{c|c} D & \Phi^*_{\eta,\eta'} \end{array}} \tag{32}$$

is in the ideal $\langle f_1(x),\ldots,f_n(x)\rangle$.

Proof of Lemma 3.2.7. First we prove that the statement holds for $x^\gamma = x_n^{\eta'}$. By [12, 3.11.11] (see also Theorem 2.2.6.(10)) we have that

$$x_n^{\eta'}\sum_{y^\beta\in\mathrm{Mon}^*(\eta)} y^\beta \mathrm{Morl}_\beta(x) \;-\; y_n^{\eta'}\sum_{y^\gamma\in\mathrm{Mon}^*(\delta-\nu)} y^\gamma\mathrm{Morl}_\gamma(x)$$

is in the ideal $\langle f_1(x),\ldots,f_n(x),f_1(y),\ldots,f_n(y)\rangle$. Therefore, there exist polynomials $q_j(x,y)$ $(1\le j\le n)$ of degree $\eta - d_j$ in y such that

$$x_n^{\eta'} \sum_{y^\beta \in \mathrm{Mon}^*(\eta)} y^\beta \mathrm{Morl}_\beta(x) - y_n^{\eta'} \sum_{y^\gamma \in \mathrm{Mon}^*(\delta-\nu)} y^\gamma \mathrm{Morl}_\gamma(x) - \sum_{j=1}^n q_j(x,y) f_j(y) \quad (33)$$

is in the ideal $\langle f_1(x), \dots, f_n(x) \rangle$. Write

$$\sum_{j=1}^n \left(q_j(x,y) f_j(y) \mod (y_n^{\eta'}) \right) = \sum_{y^\beta \in \mathrm{Mon}^*(\eta,\eta')} Q_\beta(x) y^\beta,$$

and let E be the column vector $(Q_\beta(x))_{y^\beta \in \mathrm{Mon}^*(\eta,\eta')}$. Then E is in the column space of the matrix $\Phi^*_{\eta,\eta'}$, therefore all of the maximal minors of the matrix

$$\left[\begin{array}{c|c} \mathrm{E} & \Phi^*_{\eta,\eta'} \end{array}\right] \quad (34)$$

are zero. Finally note that by (33) the maximal minors of the matrix (32) and of the matrix (34) are congruent modulo $\langle f_1(x), \dots, f_n(x) \rangle$, which proves the claim for $x^\gamma = x_n^{\eta'}$.

The proof for the $x^\gamma \in \mathrm{Mon}(\eta+1)$ case is similar, using the fact that by [12, 3.11.11] (see also Theorem 2.2.6.(10)) for all $x^\gamma \in \mathrm{Mon}(\eta+1)$ the polynomial $x^\gamma \cdot \sum_{y^\beta \in \mathrm{Mon}^*(\eta)} \mathrm{Morl}_\beta(x) y^\beta$ is in the ideal $\langle f_1(x), \dots, f_n(x), f_1(y), \dots, f_n(y) \rangle$. ■

Proof of Proposition 3.2.4. Using Lemma 3.2.7 it is easy to see that the matrix $\mathbf{J}_{\eta,\nu}(f)$ satisfy the conditions of Lemma 3.2.5, with

$$B = \left[\begin{array}{c} \Omega_{\eta,\eta'} \\ \hline \Phi_{\delta-\eta} \end{array}\right], \qquad C = \left[\begin{array}{c} \Phi^*_{\eta,\eta'} \\ \hline 0 \end{array}\right]$$

and the columns of B correspond to the monomials $\{x^\gamma x^\beta \mid |\beta| = \delta - \eta\}$ where $x^\gamma \in \mathrm{Mon}(\eta+1) \cup \{x_n^{\eta'}\}$ is fixed. Note that for any $T \subset \mathrm{Mon}(\delta-\eta)$ of cardinality $\mathcal{H}_d(\nu)$ the subresultant $\Gamma_T^{\eta,\nu}(f)$ is equal to $\det(\mathbf{E}_{\delta-\eta}) \cdot \det(\mathbf{E}_{\eta,\eta'})$ times the subdeterminant $\det(\mathbf{M}_T^{\eta,\nu})$ by Definition 3.2.1. Therefore the statement of Lemma 3.2.5 implies that $\det(\mathbf{E}_{\delta-\eta}) \cdot \det(\mathbf{E}_{\eta,\eta'})$ times the polynomials in (27) and in (28) are in the ideal $\langle f_1, \dots, f_n \rangle$. Using the fact that $\det(\mathbf{E}_{\delta-\eta}) \cdot \det(\mathbf{E}_{\eta,\eta'})$ does not depend on the coefficients of f_n and the fact that for generic polynomials

$f_1, \ldots, f_n \in \mathbf{R}[x_1, \ldots, x_n]$ we have

$$\langle f_1, \ldots, f_n \rangle = \mathcal{P} \cap \mathcal{R}$$

where $\mathcal{R}$ is an $\langle x_1, \ldots, x_n \rangle$-primary ideal and $\mathcal{P}$ is a prime ideal with $\mathbf{R} \cap \mathcal{P}$ principal generated by the projective resultant $\mathrm{Res}_d(f)$ (cf. [4, Proposition 3]), we conclude that $\det(\mathbf{E}_{\delta-\eta}) \cdot \det(\mathbf{E}_{\eta,\eta'}) \notin \mathcal{P}$, therefore the polynomials in (27) and in (28) are in the ideal $\langle f_1, \ldots, f_n \rangle$. ∎

Example 2.1.1 (cont)
To demonstrate the relevance of Proposition 3.2.4 we continue the examples by specifying our system. We constructed the specified system to have 3 common roots in the projective space:

$$\text{Roots } = \{(x = 2t, y = -t, z = -2t)\,, (x = -t, y = -t, z = t)\,, (x = t, y = -2t, z = 3t)\}\,.$$

The three polynomials are the following:

$$\begin{aligned}
\tilde{f}_1 &:= -\frac{335}{8}\,x^3 - 53\,x^2y - 66\,x^2z - 37\,xy^2 - 23\,xyz - \frac{129}{8}\,xz^2 + 82\,y^3 - 42\,y^2z - 34\,yz^2 + 31\,z^3,\\
\tilde{f}_2 &:= -76\,x^3 + 25\,x^2y - 65\,x^2z - 60\,xy^2 - 61\,xyz + 28\,xz^2 - 306\,y^3 - 289\,y^2z + 29\,yz^2 + 55\,z^3,\\
\tilde{f}_3 &:= 78\,x^2 + 94\,xy + \frac{599}{12}\,xz - 222\,y^2 - 17\,yz + \frac{995}{12}\,z^2
\end{aligned}$$

The subresultant matrix $\mathbf{J}_{2,4}(\tilde{f})$ is the following:

$$\begin{bmatrix}
-\frac{1205539}{32} & -\frac{3082633}{6} & -\frac{6252151}{32} & \frac{14269385}{48} & -\frac{4803415}{24} & -\frac{19881}{4} & \frac{3308429}{2} & \frac{35326601}{48} & \frac{46953}{4} & 0 & 78\\
-\frac{26987179}{24} & \frac{9131089}{48} & -\frac{14684647}{24} & \frac{7781405}{6} & \frac{21713761}{48} & \frac{1379793}{4} & 541718 & \frac{1188664}{3} & 272508 & 0 & 94\\
\frac{4356593}{48} & -\frac{29459477}{12} & \frac{54406673}{48} & \frac{830797}{2} & -\frac{1779307}{6} & 272508 & -1897408 & -\frac{197595}{2} & 1276774 & 0 & -222\\
-\frac{335}{8} & -53 & -66 & -37 & -23 & -\frac{129}{8} & 82 & -42 & -34 & 31 & 0\\
78 & 94 & \frac{599}{12} & -222 & -17 & \frac{995}{12} & 0 & 0 & 0 & 0 & 0\\
-76 & 25 & -65 & -60 & -61 & 28 & -306 & -289 & 29 & 55 & 0\\
0 & 78 & 0 & 94 & \frac{599}{12} & 0 & -222 & -17 & \frac{995}{12} & 0 & 0\\
0 & 0 & 78 & 0 & 94 & \frac{599}{12} & 0 & -222 & -17 & \frac{995}{12} & 0
\end{bmatrix}$$

with columns corresponding to the monomials

$$\left[x^3 \;\; x^2y \;\; x^2z \;\; xy^2 \;\; xyz \;\; xz^2 \;\; y^3 \;\; y^2z \;\; yz^2 \;\; z^3 \;\; z^2 \right].$$

Choosing $T := \{x^3, x^2y, x^2z\}$, since $\det(\mathbf{M}_T^{2,4}(\tilde{f})) \neq 0$, by Proposition 3.2.4, T "pseudo"-generates the factor space $\mathbb{Q}[x, y, z]_3/\langle \tilde{f} \rangle_3$. Therefore, the for all monomials $m \in \mathbb{Q}[x, y, z]_3 - T$, the polynomials of the form

$$\Gamma_T^{2,4} m + \epsilon_1 \Gamma^{2,4}_{(T \cup \{m\} - \{x^3\})} x^3 + \epsilon_2 \Gamma^{2,4}_{(T \cup \{m\} - \{x^2y\})} x^2 y + \epsilon_3 \Gamma^{2,4}_{(T \cup \{m\} - \{x^2z\})} x^2 z$$

are not identically zero and they are in the ideal $\langle \tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \rangle$ once multiplied by z (see (27)). These polynomials are

$$\begin{aligned} & xz^2-(3\,x^3+2\,x^2z), && z^3-(6\,x^3+7\,x^2z),\\ & yz^2-(-4\,x^2z-4\,x^3+x^2y), && y^2z-(-1/2\,x^2y+\tfrac{19}{8}\,x^3+\tfrac{23}{8}\,x^2z),\\ & xyz-(-2\,x^3-x^2y-2\,x^2z), && y^3-(-\tfrac{23}{16}\,x^3+3/4\,x^2y-\tfrac{27}{16}\,x^2z),\\ & xy^2-(\tfrac{13}{8}\,x^3+\tfrac{1}{2}\,x^2y+\tfrac{9}{8}\,x^2z). \end{aligned} \tag{35}$$

Then the common roots of the original system are also roots of the polynomials in (35). Because of the structure of the polynomials in (35), one can recover the y and z coordinates of the roots at $x=1$ by simply computing the eigenvalues of the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{13}{8} & \frac{1}{2} & \frac{9}{8} \\ -2 & -1 & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ -2 & -1 & -2 \\ 3 & 0 & 2 \end{bmatrix},$$

respectively. These matrices are the matrices of the multiplication map by y and z (respectively) modulo the dehomogenization of the polynomials in (35) at $x=1$, written in the basis $T|_{x=1}$. Their entries can be read out from the coefficients of the polynomials in (35). To see more details of this method see [16].

3.3 Subresultants and Koszul complexes

The motivation for the new definitions and technicalities of this subsection is to prove the main theorem of the paper that the Jouanolou type subresultants coincide with the Macaulay type subresultants (see Theorem 3.3.10).

In this subsection we describe the matrix $\mathbf{J}_{\eta,\nu}(f)$ from a decomposition of a Koszul-Weyman complex (cf. [8]). Comparing this complex to the complex corresponding to Macaulay type subresultant matrices in [4] and using techniques developed in [6] for the complex corresponding to Jouanolou's matrix, we will be able to prove that the determinant of our complex equals the subresultant defined earlier.

First let us fix the notation we use throughout this subsection. Let $f=(f_1,\ldots,f_n)$ be generic polynomials in $\mathbf{R}[x_1,\ldots,x_n]$ of degree $d=(d_1,\ldots,d_n)$, and let δ, ν, η and $\eta'=\eta-(\delta-\nu)$ be such that they satisfy $0\le\eta\le\delta-\eta\le\nu\le\delta$ as above. For any $0\le t'\le t\le\delta$ we define the following free $\mathbf{R}$-modules for $1<p\le n$

$$\bigwedge^p \mathrm{S}(t)^n := \left\langle \bigcup_{1\le i_1<\cdots<i_p\le n} \; \bigcup_{x^\alpha\in \mathrm{Mon}(t-\sum_{s=1}^p d_{i_s})} x^\alpha\cdot e_{i_1}\wedge\cdots\wedge e_{i_p}\right\rangle$$

$$\bigwedge^p \mathrm{S}^*(t,t')^n := \left\langle \bigcup_{1\le i_1<\cdots<i_p\le n} \; \bigcup_{y^\alpha\in \mathrm{Mon}^*(t-\sum_{s=1}^p d_{i_s},\,t')} y^\alpha\cdot e_{i_1}\wedge\cdots\wedge e_{i_p}\right\rangle.$$

The grading is given by $\deg(x^\alpha e_{i_1}\wedge\cdots\wedge e_{i_p}):=|\alpha|+d_{i_1}+\cdots+d_{i_p}$. As a convention, for $p=0$ we may write $\bigwedge^0 \mathrm{S}(t)^n:=\langle\mathrm{Mon}(t)\rangle$ and $\bigwedge^0 \mathrm{S}^*(t,t')^n:=\langle\mathrm{Mon}^*(t,t')\rangle$.

Next we consider the following two complexes of $\mathbf{R}$-modules. Fix $T \subseteq \mathrm{Mon}(\delta-\eta)$ of cardinality $\mathcal{H}_d(\nu)$. The first complex, denoted by $K^\bullet(f,\delta-\eta,T)$, is a restriction of the degree $\delta-\eta$ part of the Koszul complex:

$$\cdots \bigwedge^2 \mathrm{S}(\delta-\eta)^n \xrightarrow{\phi_{-2}^{(\delta-\eta)}(f)} \bigwedge^1 \mathrm{S}(\delta-\eta)^n \xrightarrow{\phi_T^{(\delta-\eta)}(f)} \langle \mathrm{Mon}(\delta-\eta)-T\rangle \longrightarrow 0. \tag{36}$$

We index the complex $K^\bullet(f,\delta-\eta,T)$ by $K^{-p}(f,\delta-\eta,T) = \bigwedge^p \mathrm{S}(\delta-\eta)^n$ for $1 < p \leq n$, and $K^0(f,\delta-\eta,T) = \langle \mathrm{Mon}(\delta-\eta)-T\rangle$.
The differentials of the complex $K^\bullet(f,\delta-\eta,T)$ are given by

$$\phi_{-p}^{(\delta-\eta)}(f) : \bigwedge^p \mathrm{S}(\delta-\eta)^n \to \bigwedge^{p-1} \mathrm{S}(\delta-\eta)^n \tag{37}$$

for $1 \leq p \leq n$, where $\phi_{-p}^{(t)}(f)$ is the differential of the degree t part of the Koszul-complex (cf. [4]), i.e.

$$\phi_{-p}^{(t)}(f)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} f_{i_k}(e_{i_1} \wedge \cdots \widehat{e_{i_k}} \cdots \wedge e_{i_p}). \tag{38}$$

For $p=1$ the differential $\phi_T^{(\delta-\eta)}(f)$ equals to $\phi_{-1}^{(\delta-\eta)}(f)$ with its image restricted to $\langle \mathrm{Mon}(\delta-\eta)-T\rangle$.

The second complex, denoted by $K^\bullet(f,\eta,\eta')^*$, is a restriction of the dual of the degree η part of the Koszul complex:

$$0 \longrightarrow \langle \mathrm{Mon}^*(\eta,\eta')\rangle \xrightarrow{\phi_1^{(\eta,\eta')*}(f)} \bigwedge^1 \mathrm{S}^*(\eta,\eta')^n \xrightarrow{\phi_2^{(\eta,\eta')*}(f)} \bigwedge^2 \mathrm{S}^*(\eta,\eta')^n \cdots \tag{39}$$

We index the complex $K^\bullet(f,\eta,\eta')^*$ by $K^p(f,\eta,\eta')^* = \bigwedge^p \mathrm{S}^*(\eta,\eta')^n$ for $0 < p \leq n$ and $K^0(f,\eta,\eta')^* = \langle \mathrm{Mon}^*(\eta,\eta')\rangle$.
The differentials of the complex $K^\bullet(f,\eta,\eta')^*$ are given by

$$\phi_p^{(\eta,\eta')*}(f) : \bigwedge^{p-1} \mathrm{S}^*(\eta,\eta')^n \;\to\; \bigwedge^p \mathrm{S}^*(\eta,\eta')^n$$

for $1 \leq p \leq n$, where $\phi_p^{(\eta,\eta')*}$ is the dual of the map $\phi_{-p}^{(\eta)}|_{\bigwedge^p \mathrm{S}(\eta,\eta')^n}$ restricted to $\bigwedge^{p-1} \mathrm{S}^*(\eta,\eta')^n$.

Consider the map Ω of complexes

$$\Omega : K^\bullet(f,\eta,\eta')^* \longrightarrow K^\bullet(f,\delta-\eta,T)$$

given by $\Omega_p := 0$ for $p \neq 0$ and Ω_0 equals to the map $\Omega_{\eta,\eta'}$ (defined in Definition 3.1.2) with its image restricted to $\langle \mathrm{Mon}(\delta-\eta) - T\rangle$, which will be denote by $\Omega_T^{\eta,\eta'}$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & \longrightarrow & \langle \mathrm{Mon}^*(\eta,\eta')\rangle & \xrightarrow{\phi_1^{(\eta,\eta'*}(f)} & \bigwedge^1 \mathrm{S}^*(\eta,\eta')^n & \longrightarrow \cdots \\
 & {\scriptstyle 0}\big\downarrow & & {\scriptstyle \Omega_T^{\eta,\eta'}}\big\downarrow & & {\scriptstyle 0}\big\downarrow & \\
\cdots \bigwedge^1 \mathrm{S}(\delta-\eta)^n & & \xrightarrow{\phi_T^{(\delta-\eta)}(f)} & \langle \mathrm{Mon}(\delta-\eta) - T\rangle & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

In the following definition we define the complex $M^\bullet(f,\eta,\nu,T)$ corresponding to the Jouanolou type subresultant as the mapping cone of the map Ω (cf. [7, Appendix 3]).

Definition 3.3.1 *Let* $\mathbf{R}$ *be a Noetherian UFD,* $f = (f_1,\ldots,f_n) \subset \mathbf{R}[x_1,\ldots,x_n]$ *be generic polynomials, let* δ, ν, η *such that they satisfy* $0 \leq \delta-\nu \leq \eta \leq \delta-\eta \leq \nu \leq \delta$ *and let* $\eta' = \eta - (\delta-\nu)$. *Using the above notation, define the free* $\mathbf{R}$*-modules*

$$\begin{aligned}
M^{-1} &:= \langle \mathrm{Mon}^*(\eta,\eta')\rangle \oplus \bigwedge^1 \mathrm{S}(\delta-\eta)^n, \\
M_T^1 &:= \langle \mathrm{Mon}(\delta-\eta) - T\rangle \oplus \bigwedge^1 \mathrm{S}^*(\eta,\eta')^n,
\end{aligned}$$

and for $1 < p \leq n$

$$\begin{aligned}
M^{-p} &:= \bigwedge^p \mathrm{S}(\delta-\eta)^n, \\
M^p &:= \bigwedge^p \mathrm{S}^*(\eta,\eta')^n.
\end{aligned}$$

Note that for $-n \leq p \leq -1$ *we have*

$$M^p = K^{p+1}(f,\eta,\eta')^* \oplus K^p(f,\delta-\eta,T)$$

and for $1 \leq p \leq n$ *we have*

$$M^p = K^{p-1}(f,\delta-\eta,T) \oplus K^p(f,\eta,\eta')^*$$

where $K^\bullet(f,\eta,\eta')^*$ *and* $K^\bullet(f,\delta-\eta,T)$ *are defined in (36) and (39).*

Also, using the above notation, define the maps

$$\partial_{-p} := \phi^{(\delta-\eta)}_{-(p+1)} \quad \textit{for} \quad 2 \le p \le n-1,$$
$$\partial_{-1} := 0 \oplus \phi^{(\delta-\eta)}_{-2},$$
$$\partial_0 := (\Omega^{(\eta,\eta')}_T + \phi^{(\delta-\eta)}_T) \oplus \phi^{(\eta,\eta')*}_1,$$
$$\partial_1 := 0 + \phi^{(\eta,\eta')}_2,$$
$$\partial_p := \phi^{(\eta,\eta')*}_{p+1} \quad \textit{for} \quad 2 \le p \le n-1.$$

As before, $\Omega^{(\eta,\eta')}_T + \phi^{(\delta-\eta)}_T$ denotes the map $\Omega_{(\eta,\eta')} + \phi^{(\delta-\eta)}_{-1}$ (see Definition 3.1.2 and (37)) with its image restricted to $\langle \mathrm{Mon}(\delta-\eta) - T\rangle$.

The complex $M^\bullet(f,\eta,\nu,T)$ corresponding to the Jouanolou type subresultant is defined as the following complex of free $\mathbf{R}$-modules:

$$\{0 \cdots \longrightarrow M^{-2} \xrightarrow{\partial_{-1}} M^{-1} \xrightarrow{\partial_0} M^1_T \xrightarrow{\partial_1} M^2 \longrightarrow \cdots 0\}.$$

Example 2.1.1 (cont)
This example demonstrates the possible difference between the subresultant matrices defined in Definition 3.1.2 and the matrix of the differential ∂_0 of the complex $M^\bullet(f,\eta,\nu,T)$ in Definition 3.3.1. We also show the possible difference between

$$\bigwedge^1 \mathrm{S}(t)^n = \bigoplus_{i=1}^n \langle \mathrm{Mon}(t-d_i)\cdot e_i\rangle \quad \text{and} \quad \langle \mathrm{Rep}_d(t)\rangle.$$

As before, we consider 3 generic polynomials of degrees $d = (3,3,2)$. If $0 < \eta < 5$ then Jouanolou's matrix $\mathbf{J}_\eta$ and all its subresultant matrices $\mathbf{J}_{\eta,\nu}$ are the same as the matrix of ∂_0 of the corresponding complex.

For $\eta = 0$ and $\nu = 5$ the subresultant matrix $\mathbf{J}_{0,5}$ has size 20×21 as we mentioned in a previous example. The matrix of ∂_0 of the complex $K^\bullet(f,0,5,T)$ (for any $T \subset \mathrm{Mon}(5)$, $|T| = 1$) has size 22×21. Its rows correspond to the 22 monomials:

$$\left[x^2,\, xy,\, xz,\, y^2,\, yz,\, z^2,\, x^2,\, xy,\, xz,\, y^2,\, yz,\, z^2,\, x^3,\, x^2y,\, x^2z,\, xy^2,\, xyz,\, xz^2,\, y^3,\, y^2z,\, yz^2,\, z^3\right].$$

Note that $\mathrm{Rep}_d(5)$ has the following 20 elements:

$$\left[x^5,\, x^4y,\, x^4z,\, x^3y^2,\, x^3yz,\, x^3z^2,\, y^3x^2,\, yx^2z^2,\, z^3x^2,\, y^4x,\, y^3xz,\, y^2xz^2,\, yxz^3,\, z^4x,\, y^5,\, y^4z,\, y^3z^2,\, y^2z^3,\, yz^4,\, z^5\right].$$

Dividing $\mathbf{x}^\alpha \in \mathrm{Rep}_d(5)$ by one of $\{x^3, y^3, z^2\}$ – the first one which divides $\mathbf{x}^\alpha$ – we get an injective, but not necessary surjective map of sets:

$$\varphi : \ \mathrm{Rep}(5) \ \rightarrow \ \mathrm{Mon}(2)\cdot e_1 \ \cup^* \ \mathrm{Mon}(2)\cdot e_2 \ \cup^* \ \mathrm{Mon}(3)\cdot e_3.$$

In fact, the maps Φ_5 (see (7)) and $\phi^{(5)}_{-1}$ (see (38)) are related the same way: while Φ_5 first divides $\mathbf{x}^\alpha \in \mathrm{Rep}(5)$ by the first one of $[x^3, y^3, z^2]$ which divides it, and then multiplies with the corresponding f_i, the map $\phi^{(5)}_{-1}$ simply multiplies $x^\beta \in \mathrm{Mon}(5-d_i)$ by f_i. The maps $\Phi_{t,t'}$ and $\phi^{(t,t')}_1$ relate similarly. The maps corresponding to the Bezoutian parts are exactly the same. ⊡

In the following proposition we prove that the complex $M^\bullet(f,\eta,\nu,T)$ is generically exact if the matrix $\mathbf{M}^{\eta,\nu}_T(f)$, defined in Definition 3.1.5, is non-singular.

Proposition 3.3.2 *Let $f = (f_1, \ldots, f_n) \in \mathbf{R}[x_1, \ldots, x_n]$, δ, ν, η and $\eta' = \eta - (\delta - \nu)$ be as above. Fix $T \subseteq \mathrm{Mon}(\delta - \eta)$ of cardinality $\mathcal{H}_d(\nu)$ such that $\det(\mathbf{M}_T^{\eta,\nu})(f) \neq 0$. Then the complex $M^\bullet(f, \eta, \nu, T)$ is generically exact.*

Proof.
We will prove that if $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ is any coefficient specialization of f with coefficients from some field k, and $\tilde{f}$ satisfies $\det(\mathbf{M}_T^{\eta,\nu}(\tilde{f})) \neq 0$ and $\ker(\Phi_{\delta-\nu}(\tilde{f})) = 0$, then the complex $M^\bullet(\tilde{f}, \eta, \nu, T)$ is exact. This implies the claim by our assumption that $\det(\mathbf{M}_T^{\eta,\nu})(f) \neq 0$ and because the map $\Phi_{\delta-\nu}$ is generically injective by [4].

We prove the proposition in four parts. First we prove that

$$K_1^\bullet(\tilde{f}, \delta - \eta): \quad \{0 \longrightarrow K^{-n} \cdots \xrightarrow{\phi_{-2}^{(\delta-\eta)}(\tilde{f})} \textstyle\bigwedge^1 \mathrm{S}(\delta - \eta)^n \xrightarrow{\phi_{-1}^{(\delta-\eta)}(\tilde{f})} \langle \mathrm{Mon}(\delta - \eta) \rangle\}$$

is exact, which implies the exactness of $M^\bullet(\tilde{f}, \eta, \nu, T)$ for levels $p \leq -2$. Secondly we prove that the complex

$$K_1^\bullet(\tilde{f}, \eta, \eta')^*: \quad \{\langle \mathrm{Mon}^*(\eta, \eta') \rangle \xrightarrow{\phi_1^{(\eta,\eta')*}(\tilde{f})} \textstyle\bigwedge^1 \mathrm{S}^*(\eta, \eta')^n \xrightarrow{\phi_2^{(\eta,\eta')*}(\tilde{f})} \cdots K^n \longrightarrow 0\}$$

is exact, which implies the exactness of $M^\bullet(\tilde{f}, \eta, \nu, T)$ for $p \geq 2$. Then we separately prove that $M^\bullet(\tilde{f}, \eta, \nu, T)$ is exact at the $p = 1$ and $p = -1$ levels.

The exactness of $K_1^\bullet(\tilde{f}, \delta - \eta)$ follows from $\ker(\Phi_{\delta-\eta}) = 0$ by $\det(\mathbf{M}_T^{\eta,\nu}(\tilde{f})) \neq 0$, and from [4].

The exactness of the complex $K_1^\bullet(\tilde{f}, \eta, \eta')^*$ is equivalent to the exactness of the dual complex $K_1^\bullet(\tilde{f}, \eta, \eta')$. Consider the short exact sequence of complexes

$$0 \longrightarrow K_1^\bullet(\tilde{f}, \delta - \nu) \xrightarrow{\iota} K_1^\bullet(\tilde{f}, \eta) \xrightarrow{\pi} K_1^\bullet(\tilde{f}, \eta, \eta') \longrightarrow 0, \qquad (40)$$

where ι_p is the multiplication map by $x_n^{\eta'}$ and $\pi - p$ is a projection for $0 \leq p \leq n$. Then ι and π commute with the differentials of the complexes, which can be checked easily.

We prove that the complexes $K_1^\bullet(\tilde{f}, \delta - \nu)$ and $K_1^\bullet(\tilde{f}, \eta)$ are exact. The exactness of $K_1^\bullet(\tilde{f}, \delta - \nu)$ follows from $\ker(\Phi_{\delta-\nu}(\tilde{f})) = 0$ and [4].

Moreover, if $\det(\mathbf{M}_T^{\eta,\nu}(\tilde{f})) \neq 0$ then both $\ker(\Phi_{\delta-\nu}(\tilde{f})) = 0$ and $\ker(\Phi_{\eta,\eta'}(\tilde{f})) = 0$, which implies that $\ker(\Phi_\eta(\tilde{f})) = 0$. Therefore, by [4] the complex $K_1^\bullet(\tilde{f}, \eta)$ is exact.

Also, since $\ker(\Phi_{\eta,\eta'}(\tilde{f})) = 0$, then using Lemma 3.1.3 we can choose the set $S_2 \subset \mathrm{Mon}(\eta)$ such that $x_n^{\eta'} \cdot S_1 \subset S_2$. Therefore, if we define $S_3 := S_2 - x_n^{\eta'} \cdot S_1$, then we have that the map $\phi_{S_3}^{\eta,\eta'}: \bigwedge^1 \mathrm{S}(\eta, \eta')^n \mapsto \langle \mathrm{Mon}(\eta, \eta') - S_3 \rangle$ is also surjective. This implies that the short sequence of the 0-th cohomologies of the complexes in (40) is exact:

$$0 \longrightarrow \frac{\langle \mathrm{Mon}(\delta-\nu)\rangle}{\mathrm{Im}\ \Phi_{\delta-\nu}(\tilde f)} \xrightarrow{\ \iota\ } \frac{\langle \mathrm{Mon}(\eta)\rangle}{\mathrm{Im}\ \Phi_{\eta}(\tilde f)} \xrightarrow{\ \pi\ } \frac{\langle \mathrm{Mon}(\eta,\eta')\rangle}{\mathrm{Im}\ \Phi_{\eta,\eta'}(\tilde f)} \longrightarrow 0.$$

Now using the long exact sequence of the cohomologies (see [7, A3.8]) of the complexes in (40), we deduce that $K_1^\bullet(\tilde f,\eta,\eta')$ is also exact.

To prove that $M^\bullet(\tilde f,\eta,\nu,T)$ is exact at $p=-1$ we show that

$$\ker(\partial_0) = \ker(\phi_{-1}^{(\delta-\eta)}) = \mathrm{Im}(\phi_{-2}^{(\delta-\eta)}) = \mathrm{Im}(\partial_{-1}). \tag{41}$$

Recall that $\partial_0 = (\Omega_T^{(\eta,\eta')} + \phi_T^{(\delta-\eta)}) \oplus \phi_1^{(\eta,\eta')*}$ where $\Omega_T^{(\eta,\eta')} + \phi_T^{(\delta-\eta)}$ denotes the map $\Omega_{\eta,\eta'} + \phi_{-1}^{(\delta-\eta)}$ with its image restricted to $\langle \mathrm{Mon}(\delta-\eta) - T\rangle$. Assume that

$$\sum_{y^\alpha \in \mathrm{Mon}^*(\eta,\eta')} a_\alpha y^\alpha + \sum_{x^\beta \in \bigoplus_i \mathrm{Mon}(\delta-\eta-d_i)} b_\beta x^\beta \in \ker\,(\Omega_T^{(\eta,\eta')} + \phi_T^{(\delta-\eta)}) \oplus \phi_1^{(\eta,\eta')*}.$$

Then $a_\alpha = 0$ for all $y^\alpha \in \mathrm{Mon}^*(\eta,\eta')$, otherwise we would get a non-trivial combination of the rows corresponding to $\mathrm{Mon}^*(\eta,\eta')$ of the matrix $\mathbf{M}_T^{\eta,\nu}(\tilde f)$ which combination is in the image $\Phi_T^{(\delta-\eta)}(\tilde f)$ (the image of $\Phi_{\delta-\eta}$ in (7) restricted to $\mathrm{Mon}(\delta-\eta)-T$). This would imply that the matrix $\mathbf{M}_T^{\eta,\nu}(\tilde f)$ is singular, a contradiction. Therefore $\ker(\partial_0) = \ker(\phi_T^{(\delta-\eta)})$. But since $\det(\mathbf{M}_T^{\eta,\nu}(\tilde f)) \neq 0$, we have that $\ker(\phi_T^{(\delta-\eta)}) = \ker(\phi_{-1}^{(\delta-\eta)})$, which proves that $\ker(\partial_0) = \ker(\phi_{-1}^{(\delta-\eta)})$ and the rest of (41) follows from the exactness of $K_1^\bullet(\tilde f, \delta-\nu)$.

Finally, we prove that $M^\bullet(\tilde f,\eta,\nu,T)$ is exact at $p=1$. By $\det(\mathbf{M}_T^{\eta,\nu}(\tilde f)) \neq 0$ the map $(\Omega_T^{(\eta,\eta')} + \phi_T^{(\delta-\eta)}) \oplus \Phi_{\eta,\eta'}^*$ is surjective. Therefore the image of ∂_0 is generated by $\mathrm{Mon}(\delta-\eta) \cup \mathrm{Rep}^*(\eta,\eta')$. This implies that the image of $\phi_1^{(\eta,\eta')*}(\tilde f)$ is generated by $\mathrm{Rep}^*(\eta,\eta')$, and by the exactness of $K_1^\bullet(\tilde f,\eta,\eta')^*$ we have that $\mathrm{Im}(\phi_1^{(\eta,\eta')*}(\tilde f)) = \ker(\phi_2^{(\eta,\eta')*}(\tilde f))$, therefore $\mathrm{Im}(\partial_0(\tilde f)) = \langle \mathrm{Mon}(\delta-\eta)\rangle \oplus \ker(\phi_2^{(\eta,\eta')*}(\tilde f)) = \ker(\partial_1(\tilde f))$ which proves the exactness at level $p=1$. ∎

Remark 3.3.3 In the proof above we asserted that $\ker(\Phi_{\eta,\eta'}) = 0$ and $\ker(\Phi_{\delta-\nu}) = 0$ implies that $\ker(\Phi_\eta) = 0$. The other direction is not necessary true: $\ker(\Phi_\eta) = 0$ and $\ker(\Phi_{\delta-\nu}) = 0$ does not imply that $\ker(\Phi_{\eta,\eta'}) = 0$. A counter example is $\tilde f = (x_1^{d_1},\dots,x_n^{d_n})$. We note that the converse of the statement of Proposition 3.3.2 is also true.

Since the complex $M^\bullet(f,\eta,\nu,T)$ is generically exact, the following definition is meaningful:

Definition 3.3.4 *Let $f=(f_1,\ldots,f_n)\subset \mathbf{R}[x_1,\ldots,x_n]$ be homogeneous polynomials of degrees $d=(d_1,\ldots,d_n)$ and let δ, ν, η and $\eta'=\eta-(\delta-\nu)$ be as above. Let $T\subseteq \mathrm{Mon}(\delta-\eta)$ be of cardinality $\mathcal{H}_d(\nu)$ such that $\det(\mathbf{M}_T^{\eta,\nu})\neq 0$. Let $\mathbf{K}$ be the fraction field of $\mathbf{R}$. We denote by $D_T^{\eta,\nu}(f)$ the determinant of the based complex of $\mathbf{K}$-vector spaces $M^\bullet(f,\eta,\nu,T)\otimes_{\mathbf{R}}\mathbf{K}$ (cf. [8, Appendix A]).*

In the next proposition we prove that $D_T^{\eta,\nu}(f)$ is equal to the corresponding subresultant $\Gamma_T^{\eta,\nu}(f)$ defined in Definition 3.2.1.

Proposition 3.3.5 *Let $f=(f_1,\ldots,f_n)$ be generic, and let δ, ν, η and $\eta'=\eta-(\delta-\nu)$ be as above. As before, let $\mathbf{E}_{\delta-\eta}$ denote the submatrix of $\Phi_{\delta-\eta}$ with rows and columns corresponding to monomials in $\mathrm{Dod}_d(\delta-\eta)$, and let $\mathbf{E}_{\eta,\eta'}$ denote the submatrix of $\Phi^*_{\eta,\eta'}$ with rows and columns corresponding to $\mathrm{Dod}_d(\eta)\cap\mathrm{Rep}_d(\eta,\eta')$. Then for any $T\subseteq\mathrm{Mon}(\delta-\eta)$ of cardinality $\mathcal{H}_d(\nu)$ such that $\det(\mathbf{M}_T^{\eta,\nu})\neq 0$ we have*

$$D_T^{\eta,\nu}=\frac{\det(\mathbf{M}_T^{\eta,\nu})}{\det(\mathbf{E}_{\delta-\eta})\det(\mathbf{E}_{\eta,\eta'})}, \tag{42}$$

and the denominator is not identically zero.

Proof:
First note that the matrix $\mathbf{M}_T^{\eta,\nu}$ is a submatrix of the matrix of ∂_0, and it corresponds to a *decomposition* (see e.g. [4]) of the complex $M^\bullet(f,\nu,\eta,T)$. Therefore, by the definition of the determinant of a complex, their ratio $\Delta:=\frac{\det(\mathbf{M}_T^{\eta,\nu})}{D_T^{\eta,\nu}}$ is the product of the determinants of the two complexes

$$\begin{aligned}
K_2^\bullet(f,\delta-\eta):\quad &\{0\longrightarrow\cdots{\textstyle\bigwedge^2}\mathrm{S}(\delta-\eta)^n\xrightarrow{\phi_{-2}^{(\delta-\eta)}(f)}S_d(\delta-\eta)\longrightarrow 0\},\\
K_2^\bullet(f,\eta,\eta')^*:\quad &\{0\longrightarrow S_d^*(\eta,\eta')\xrightarrow{\phi_2^{(\eta,\eta')*}(f)}{\textstyle\bigwedge^2}\mathrm{S}^*(\eta,\eta')^n\cdots\longrightarrow 0\}.
\end{aligned} \tag{43}$$

where $S_d(\delta-\eta)$ and $S_d(\eta,\eta')$ corresponds to decompositions of the $\mathbf{R}$-modules $\bigwedge^1\mathrm{S}(t)^n$ and $\bigwedge^1\mathrm{S}^*(t,t')^n$ in the complexes $K^\bullet(f,\delta-\eta)$ and $K^\bullet(f,\eta,\eta')^*$ respectively (see (36) and (39)). For example, similarly to [4], we can choose the decomposition of $\bigwedge^1\mathrm{S}(t)^n$ for any $t\geq 0$ to be

$$\bigwedge^1\mathrm{S}(t)^n=S_d(t)+\langle\mathrm{Rep}'_d(t)\rangle$$

where $\mathrm{Rep}'_d(t)=\{x^\alpha/x_{i(\alpha)}^{d_{i(\alpha)}}\mid x^\alpha\in\mathrm{Rep}_d(t)\}$. Also, we can decompose $\bigwedge^1\mathrm{S}^*(\eta,\eta')^n$

for $0 \leq t' \leq t$ into

$$\bigwedge^1 \mathrm{S}^*(t,t')^n = S_d^*(t,t') + \langle \mathrm{Rep}_d'(t,t')^* \rangle$$

where $\mathrm{Rep}_d'(t,t')^* = \{y^\beta / y_{i(\beta)}^{d_{i(\beta)}} \mid y^\beta \in \mathrm{Rep}_d^*(t,t')\}$.

Clearly neither of the complexes in (43) depend on the choice of T. Therefore it is enough to prove the claim for a fixed T of cardinality $\mathcal{H}_d(\nu)$ such that $\det(\mathbf{M}_T^{\eta,\nu}) \neq 0$.

It follows from [4] that the determinant of $K_2^\bullet(f, \delta - \eta)$ is $\det(\mathbf{E}_{\delta-\eta}(f))$, and it is not identically zero. On the other hand, as we have seen it in the proof of Proposition 3.3.2, the complexes $K_2^\bullet(f, \delta - \nu)$ and $K_2^\bullet(f, \eta)$ are generically exact, and by [4], their determinants are $\det(\mathbf{E}_{\delta-\nu}(f))$ and $\det(\mathbf{E}_\eta(f))$ respectively, and neither of them is identically zero. Using [8, Appendix A, Lemma 5] and the exact sequence of complexes in (40), we get that the determinant of $K_2^\bullet(f, \eta, \eta')$ is the ratio $\frac{\det(\mathbf{E}_\eta(f))}{\det(\mathbf{E}_{\delta-\nu}(f))}$. But by Lemma 3.1.3 $\det(\mathbf{E}_\eta(f)) = \det(\mathbf{E}_{\eta,\eta'}(f)) \cdot \det(\mathbf{E}_{\delta-\nu}(f))$, therefore,

$$D_T^{\eta,\nu}(f) = \frac{\det(\mathbf{M}_T^{\eta,\nu})}{\Delta}(f) = \frac{\det(\mathbf{M}_T^{\eta,\nu})}{\det(\mathbf{E}_{\delta-\eta}) \frac{\det(\mathbf{E}_\eta)}{\det(\mathbf{E}_{\delta-\nu})}}(f) = \frac{\det(\mathbf{M}_T^{\eta,\nu})}{\det(\mathbf{E}_{\delta-\eta}) \det(\mathbf{E}_{\eta,\eta'})}(f). \quad \blacksquare$$

Before we state the next corollary we include the definition of *multiplicity* of a finitely generated $\mathbf{R}$-module along a prime ideal $\mathfrak{p} \subset \mathbf{R}$ from [8].

Definition 3.3.6 *Let $\mathbf{R}$ be a Noetherian UFD and $\mathfrak{p} \subset \mathbf{R}$ be a prime ideal. Denote by $\mathbf{R}_\mathfrak{p}$ the localization of $\mathbf{R}$ at $\mathfrak{p}$ with maximal ideal $\mathfrak{m}_\mathfrak{p}$ and with associated field $k_\mathfrak{p} = \mathbf{R}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$. If M' is a finitely generated $\mathbf{R}_\mathfrak{p}$-module, then we say that M' has finite length if there exists $i \gg 0$ such that $\mathfrak{m}_\mathfrak{p}^i \cdot M' = 0$ and for such M' the multiplicity of M' is defined*

$$\mathrm{mult}_{\mathfrak{m}_\mathfrak{p}}(M') = \sum_i \dim_{k_\mathfrak{p}} \mathfrak{m}_\mathfrak{p}^i \cdot M' / \mathfrak{m}_\mathfrak{p}^{i+1} \cdot M'.$$

For a finitely generated $\mathbf{R}$-module M, denote $M_\mathfrak{p} = M \otimes \mathbf{R}_\mathfrak{p}$. Then we define the multiplicity of M at $\mathfrak{p}$ by

$$\mathrm{mult}_\mathfrak{p}(M) = \begin{cases} \mathrm{mult}_{\mathfrak{m}_\mathfrak{p}}(M_\mathfrak{p}) & \text{if } M_\mathfrak{p} \text{ has finite length} \\ 0 & \text{otherwise} \end{cases}$$

We also included the definition of the order of a polynomial with respect to a prime:

Definition 3.3.7 *or $p \in \mathbf{R}$ prime and for $F \in \mathbf{R}$ we define $\mathrm{ord}_p(F)$ to be the highest power i such that p^i divides F.*

Corollary 3.3.8 *Let $f = (f_1, \ldots, f_n) \subset \mathbf{R}[x_1, \ldots, x_n]$ be generic polynomials where $\mathbf{R}$ is a Noetherian UFD. Let $d = (d_1, \ldots, d_n)$, δ, ν, η and η' be as above. Let $T \subset \mathrm{Mon}(\delta - \eta)$ be of cardinality $\mathcal{H}_d(\nu)$ such that $\Gamma_T^{\eta;\nu}(f) \not\equiv 0$. Then for any prime element $p \in \mathbf{R}$ we have*

$$\mathrm{ord}_p(\Gamma_T^{\eta;\nu}(f)) = \sum_{i=-n}^{n-1} (-1)^i \mathrm{mult}_{\langle p \rangle}(H^i(M^\bullet(f, \eta, \nu, T)))$$

where $H^i(M^\bullet(f, \eta, \nu, T))$ denotes the cohomology module $\ker(\partial_{i+1})/\mathrm{Im}(\partial_i)$ (defined in Definition 3.3.1). ■

The next lemma is used in Theorem 3.3.10 at the end of the paper. It can be viewed as the converse of Proposition 3.2.4. Its proof is an easy consequence [12, 3.11.11 and 3.8.2.9], and we leave the details to the reader.

Lemma 3.3.9 *Let $f = (f_1, \ldots, f_n)$ be generic polynomials in $\mathbf{R}[x_1, \ldots, x_n]$, and let δ, ν, η and $\eta' = \eta - (\delta - \nu)$ be as above. Let $p \in \mathbf{R}[x_1, \ldots, x_n]$ be a degree $\delta - \eta$ polynomial such that*

$$x_n^{\eta'} \cdot p(x) \in \langle f_1(x), \ldots, f_n(x) \rangle_\nu.$$

Then there exist $a_\beta \in \mathbf{R}$ for all $x^\beta \in \mathrm{Mon}(\eta, \eta')$ such that

$$\mathrm{Res}_d(f) p(x) - \sum_{x^\beta \in \mathrm{Mon}(\eta, \eta')} a_\beta \mathrm{Morl}_\beta(x)$$

is in $\langle f_1(x), \ldots, f_n(x) \rangle_{\delta - \eta}$.
Moreover, the if we denote by $\mathbf{a}$ the vector $(a_\beta)_{x^\beta \in \mathrm{Mon}(\eta, \eta')}$, and by $\mathbf{b}$ the vector of the coefficients of the polynomial $(x^\rho f_i(x) \mod x_n^{\eta'}) \in \langle \mathrm{Mon}(\eta, \eta') \rangle$, then $\mathbf{a}^T \mathbf{b} = 0$ for any $x^\rho \in \mathrm{Mon}(\eta - d_i)$ and $1 \leq i \leq n$.

Finally, we prove the main theorem of the paper, that the subresultants defined using Jouanolou's matrices are the same as the ones constructed from Macaulay type matrices.

Theorem 3.3.10 *Let $\mathbf{R}$ be a Noetherian UFD, $f = (f_1, \ldots, f_n) \subset \mathbf{R}[x_1, \ldots, x_n]$ be generic polynomials, let δ, ν, η such that they satisfy $0 \leq \delta - \nu \leq \eta \leq \delta - \eta \leq \nu \leq \delta$ and let $\eta' = \eta - (\delta - \nu)$. For any set $T \subset \mathrm{Mon}(\delta - \eta)$ of cardinality $\mathcal{H}_d(\nu)$ define the set $S := x_n^{\eta'} \cdot T \subset \mathrm{Mon}(\nu)$. If $\Gamma_T^{\eta;\nu}(f)$ is not identically zero then*

$$\Gamma_T^{\eta,\nu}(f) = \Delta_S^\nu(f),$$

where $\Delta_S^\nu(f)$ is the Macaulay type subresultant (see [4] and subsection 2.1). Note that this also implies that $\Gamma_T^{\eta,\nu}(f)$ does not depend on η.

Proof: We shall prove that for any prime element $p \in \mathbf{R}$

$$\mathrm{ord}_p(\Gamma_T^{\eta,\nu}(f)) \leq \mathrm{ord}_p(\Delta_S^\nu(f)). \tag{44}$$

Then (44) implies the claim of the theorem since by Proposition 3.2.3 we have that for each $1 \leq i \leq n$

$$\deg_{f_i}(\Gamma_T^{\eta,\nu}(f)) = \deg_{f_i}(\Delta_S^\nu(f)),$$

therefore $\Gamma_T^{\eta,\nu}(f)$ must be equal to $\Delta_S^\nu(f)$.

For $p = \mathrm{Res}_d(f)$ the inequality (44) holds since

$$\mathrm{ord}_{\mathrm{Res}_d(f)}(\Gamma_T^{\eta,\nu}(f)) = \mathrm{ord}_{\mathrm{Res}_d(f)}(\Delta_S^\nu(f)) = 0$$

by comparing degrees.

Let us assume that $p \neq \mathrm{Res}_d(f)$. To prove (44) first note that both sides of (44) are of local nature, so we can assume that $\mathbf{R}$ is a local ring with maximal ideal $\langle p \rangle$ and associated field $\mathbf{k} = \mathbf{R}/\langle p \rangle$. To simplify the notation, for any $\mathbf{R}$-module M and for any $i \geq 0$ we denote the $\mathbf{k}$-vectorspace $p^i \cdot M \;/\; p^{i+1} \cdot M$ by

$$\mathfrak{p}_i(M) := p^i \cdot M \;/\; p^{i+1} \cdot M.$$

Also, for a matrix $\mathbf{M} = (m_{s,t})_{s,t=1}^{k,l} \in \mathbf{R}^{k,l}$ we denote by $\mathfrak{p}_i(\mathbf{M})$ the matrix

$$\mathfrak{p}_i(\mathbf{M}) = (p^i m_{s,t} \mod p^{i+1})_{s,t=1}^{k,l} \in \mathfrak{p}_i(\mathbf{R})^{k,l}.$$

Using Definition 3.2.1 and [3] we have that

$$\begin{aligned} \mathrm{ord}_p(\Gamma_T^{\eta,\nu}(f)) &= \mathrm{ord}_p(\det(\mathbf{M}_T^{\eta,\nu})) - \mathrm{ord}_p(\det(\mathbf{E}_{\delta-\eta})) - \mathrm{ord}_p(\det(\mathbf{E}_{\eta,\eta'})) \\ \mathrm{ord}_p(\Delta_S^\nu(f)) &= \mathrm{ord}_p(\det(\mathbf{M}_S^{\delta-\nu,\nu})) - \mathrm{ord}_p(\det(\mathbf{E}_\nu)), \end{aligned}$$

where $\mathbf{M}_S^{\delta-\nu,\nu}$ denotes the submatrix of $\mathbf{J}_{\delta-\nu,\nu}(f)$ with columns not belonging to S, and $\mathbf{J}_{\delta-\nu,\nu}(f)$ is the Macaulay type subresultant matrix of degree ν (as a special case of Jouanolou type subresultant matrices). Moreover, by [8, Appendix A, Theorem 30] and Definition 3.3.6 we have that

$$\mathrm{ord}_p(\det(\mathbf{M}_T^{\eta,\nu})) = \mathrm{mult}_{\langle p \rangle}(\mathrm{Coker}\ \mathbf{M}_T^{\eta,\nu}) = \sum_{i \geq 0} \dim_{\mathbf{k}} \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{M}_T^{\eta,\nu}),$$

where $\mathbf{M}_T^{\eta,\nu}$ also denotes the $\mathbf{k}$-linear map corresponding to the rows of the matrix $\mathbf{M}_T^{\eta,\nu}$. Similar equations hold for $\mathrm{ord}_p(\det(\mathbf{M}_S^{\delta-\nu,\nu}))$ and for $\mathrm{ord}_p(\det(\mathbf{E}_t)$ for any $t > 0$.

To simplify the notation we denote

$$k := \dim_{\mathbf{k}} \ \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{E}_{\delta-\eta}) \geq 0 \tag{45}$$

and

$$l := \dim_{\mathbf{k}} \ \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{E}_{\eta,\eta'}) \geq 0. \tag{46}$$

Fix some $i \geq 0$. Let $B_1 \subset \mathrm{Rep}_d^*(\eta,\eta')$ be such that the corresponding columns of $\mathfrak{p}_i(\Phi^*_{\eta,\eta'})$ form a basis (over $\mathbf{k}$) for the column-space of $\mathfrak{p}_i(\Phi^*_{\eta,\eta'})$ (for $\Phi^*_{\eta,\eta'}$ see Definition 3.1.2). Let $B_2 \subset \mathrm{Mon}(\delta-\eta) - T$ be such that the columns of $\mathfrak{p}_i(\mathbf{M}_T^{\eta,\nu})$ corresponding to $B_1 \cup B_2$ form a basis for the column space of $\mathfrak{p}_i(\mathbf{M}_T^{\eta,\nu})$. Let $C_1 := \mathrm{Rep}_d^*(\eta,\eta') - B_1$ and let $C_2 := \mathrm{Mon}(\delta-\eta) - T - B_2$. Then

$$|C_1| + |C_2| = \dim_{\mathbf{k}} \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{M}_T^{\eta,\nu}).$$

The claim (44) follows if we prove that for any $i \geq 0$

$$|C_1| + |C_2| - k - l \ \leq \dim_{\mathbf{k}} \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{M}_S^{\delta-\nu,\nu}) - \dim_{\mathbf{k}} \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{E}_\nu)$$

which, by [4], is equivalent to

$$|C_1| + |C_2| - k - l \ \leq \dim_{\mathbf{k}} \mathfrak{p}_i\langle \mathrm{Mon}(\nu)\rangle - \dim_{\mathbf{k}} \left(\mathfrak{p}_i\langle S\rangle + \mathfrak{p}_i\langle f_1,\ldots,f_n\rangle_\nu\right). \tag{47}$$

We prove (47) in two steps:
<u>Claim 1:</u> There exists a subspace $V_1 \subset \mathfrak{p}_i\langle\overline{\mathrm{Mon}}(\nu,\eta')\rangle$ (see Definition 3.1.1) such that

$$\dim_{\mathbf{k}}(V_1) \geq |C_2| - k, \quad \text{and} \quad V_1 \ \cap \ (\mathfrak{p}_i\langle S\rangle \oplus \mathfrak{p}_i\langle f_1,\ldots,f_n\rangle_\nu) = \{0\}. \tag{48}$$

<u>Claim 2:</u> There exists a subspace $V_2 \subset \mathfrak{p}_i\langle \mathrm{Mon}(\nu,\eta')\rangle$ such that

$$\dim_{\mathbf{k}}(V_2) \geq |C_1| - l, \quad \text{and} \quad V_2 \ \cap \ (\mathfrak{p}_i\langle S\rangle \oplus \mathfrak{p}_i\langle f_1,\ldots,f_n\rangle_\nu) = \{0\}. \tag{49}$$

Clearly, Claim 1 and Claim 2 imply (47), thus also the claim of the theorem. We will prove Claim 1 and Claim 2 separately using Lemma 3.3.11 and Lemma 3.3.12 below. ■

Claim 1 follows from the following lemma:

Lemma 3.3.11 *Using the notations and assumptions introduced in the proof of Theorem 3.3.10, define the* $\mathbf{k}$*-space*

$$V := (\mathfrak{p}_i\langle S\rangle \oplus \mathfrak{p}_i\langle f_1,\dots,f_n\rangle_\nu) \;\cap\; \mathfrak{p}_i\langle x_n^{\eta'}\cdot C_2\rangle. \tag{50}$$

Then V *has dimension at most* k*.*

Proof of Lemma 3.3.11: By the definition of V in (50), for any element $x_n^{\eta'}\cdot q(x)\in V$ there exists $c_\gamma\in\mathbf{k}$ for all $x^\gamma\in T$ such that

$$x_n^{\eta'}\cdot q(x)+\sum_{x^\gamma\in T} c_\gamma\cdot x_n^{\eta'}\cdot x^\gamma \in \mathfrak{p}_i\langle f_1,\dots,f_n\rangle_\nu.$$

Note that we used the fact $S = x_n^{\eta'}\cdot T$. Therefore, we can define the natural projection

$$\pi_1 : V\to \mathfrak{p}_i\langle f_1,\dots,f_n\rangle_\nu\cap\mathfrak{p}_i\langle x_n^{\eta'}\cdot C_2\rangle$$

with fibers in $\mathfrak{p}_i\langle S\rangle$. Note that π_1 is injective on $\mathfrak{p}_i\langle x_n^{\eta'}C_2\rangle$ since $x_n^{\eta'}C_2\cap S=\emptyset$. Let $r(x)\in\mathbf{R}[x]$ be an inverse image of some element of $\frac{1}{x_n^{\eta'}}\cdot\pi_1(V)$, i.e.

$$r(x)\equiv q(x)+\sum_{x^\gamma\in T} c_\gamma\cdot x^\gamma \mod p^{i+1}$$

for some $x_n^{\eta'}q(x)\in V$. Since $x_n^{\eta'}r(x)\in\langle f_1,\dots,f_n\rangle_\nu$, applying Lemma 3.3.9 we get that there exist $a_\beta\in\mathbf{R}$ for all $x^\beta\in\mathrm{Mon}(\eta,\eta')$ such that

$$\mathrm{Res}_d(f)r(x)+\sum_{x^\beta\in\mathrm{Mon}(\eta,\eta')} a_\beta\mathrm{Morl}_\beta(x) \quad\in\quad \langle f_1,\dots,f_n\rangle_{\delta-\eta}.$$

Moreover, by the second claim of Lemma 3.3.9, we have that the matrix product

$$(a_\beta)^T_{y^\beta\in\mathrm{Mon}^*(\eta,\eta')}\cdot\Phi^*_{\eta,\eta'}=0.$$

Thus the rows of the subresultant matrix $\mathbf{J}_{\eta,\nu}(f)$ span the vector corresponding to the coefficients of $\mathrm{Res}_d(f)r(x)$ plus some polynomial in $\langle f_1,\dots,f_n\rangle_{\delta-\eta}$. Therefore, using the fact that $\mathrm{Res}_d(f)$ is a unit in $\mathbf{R}$ by assumption, there exist a projection

$$\pi_2 : \frac{1}{x_n^{\eta'}}\cdot\pi_1(V)\to\mathfrak{p}_i\langle f_1,\dots,f_n\rangle_{\delta-\eta}$$

such that the fibers of π_2 are in $\mathfrak{p}_i(\mathrm{Im}\;\Omega_{\eta,\eta'}\oplus\Phi^*_{\eta,\eta'})$. Note that π_2 is injective on $\mathfrak{p}_i\langle C_2\rangle$ since elements of $\mathfrak{p}_i\langle C_2\rangle$ are not in $\mathfrak{p}_i(\mathrm{Im}\;\Omega_{\eta,\eta'}\oplus\Phi^*_{\eta,\eta'})$.

Since the fibers of the projection $\frac{1}{x_n^{\eta'}} \cdot \pi_1 \circ \pi_2$ are in $\mathfrak{p}_i(\mathrm{Im}\ \Omega_T^{\eta,\eta'} \oplus \Phi^*_{\eta,\eta'})$, therefore we must have

$$(\frac{1}{x_n^{\eta'}} \cdot \pi_1 \circ \pi_2)(V) \ \cap\ \mathfrak{p}_i(\mathrm{Im}\ \Phi_{\delta-\eta}) = \{0\},$$

otherwise there exist non-zero elements of $\mathfrak{p}_i\langle C_2\rangle$ which are in $\mathfrak{p}_i(\mathrm{Im}\ \mathbf{M}_T^{\eta,\nu})$, contradicting the definition of C_2. Since

$$\dim_{\mathbf{k}} \mathfrak{p}_i(\langle f_1, \ldots, f_n\rangle_{\delta-\eta}) - \dim_{\mathbf{k}} \mathfrak{p}_i(\mathrm{Im}\ \Phi_{\delta-\eta}) = \dim_{\mathbf{k}}\ \mathfrak{p}_i(\mathrm{Coker}\ \mathbf{E}_{\delta-\eta}) = k,$$

therefore $(\frac{1}{x_n^{\eta'}} \cdot \pi_1 \circ \pi_2)(V)$ has dimension at most k . Using the injectivity of π_1 and π_2 this implies that V has dimension at most k. This concludes the proof of Lemma 3.3.11. ■

Claim 2 follows from the following lemma by taking $V_2 := \mathfrak{p}_i(\mathrm{Coker}\ \phi_1^{(\nu,\eta')})$ defined below:

Lemma 3.3.12 *Using the notations and assumptions introduced in the proof of Theorem 3.3.10, the map*

$$\begin{aligned} \phi_1^{(\nu,\eta')} : \bigoplus_{i=1}^{n} \mathrm{Mon}(\nu - d_i, \eta') &\to \mathrm{Mon}(\nu, \eta') \\ (g_1, \ldots, g_n) &\mapsto \sum_{i=1}^{n} f_i g_i \quad \mod x_n^{\eta'} \end{aligned}$$

satisfies

$$\dim_{\mathbf{k}}\ \mathfrak{p}_i(\mathrm{Coker}\ \phi_1^{(\nu,\eta')}) \geq |C_1| - l. \tag{51}$$

Proof of Lemma 3.3.12: First note that since $0 \leq \delta - \nu \leq \eta \leq \delta - \eta \leq \nu \leq \delta$, we have that $\#\mathrm{Rep}_d(\nu, \eta') \geq \#\mathrm{Mon}(\nu, \eta')$ which implies that $\phi_1^{(\nu,\eta')}$ is generically surjective.

Recall that $C_1 \subset \mathrm{Rep}^*_d(\eta, \eta')$ was chosen to be a basis for $\mathfrak{p}_i(\mathrm{Coker}\ \Phi^*_{\eta,\eta'})$. Considering the dual of the map $\Phi^*_{\eta,\eta'}$ we get that C_1 corresponds to a basis of $\mathfrak{p}_i(\ker\ \Phi_{\eta,\eta'})$. Taking into account the definition of l in (46) we get that the first cohomology of the Koszul complex $K^\bullet((f_1, \ldots, f_n, x_n^{\eta'}), \eta)$ of the $n+1$ polynomials $f_1, \ldots, f_n, x_n^{\eta'}$ satisfies

$$\dim_{\mathbf{k}} \mathfrak{p}_i\ H^1(K^\bullet((f_1, \ldots, f_n, x_n^{\eta'}), \eta)) = |C_1| - l. \tag{52}$$

We can rewrite the claimed inequality (51) as well:

$$\dim_{\mathbf{k}}\ \mathfrak{p}_i\ H^0(K^\bullet((f_1,\ldots,f_n,x_n^{\eta'}),\nu)) \geq |C_1| - l. \tag{53}$$

Since $p^i \in R$ defines a hypersurface in the coefficient space of $f_1,\ldots,f_n$, we can assume without loss of generality (maybe after permutation of indeces) that $f_1,\ldots,f_{n-1}$ are generic polynomials. Define the system of polynomials

$$f' := (f_1,\ldots,f_{n-1},x_n^{\eta'})$$

with degrees $d' = (d_1,\ldots,d_{n-1},\eta')$. By the genericity of $f_1,\ldots,f_{n-1}$ we can assume that for any $t \geq 0$ the cohomologies of the Koszul complex of f' satisfies

$$\mathfrak{p}_i\ H^j(K^\bullet(f',t)) = 0\ \ \forall\, j \geq 1.$$

Next we consider the mapping cone of the map of complexes

$$\psi_{f_n} : K^\bullet(f',t-d_n) \to K^\bullet(f',t)$$

defined by the multiplication by f_n. We have the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \bigwedge^2 \mathrm{S}'(t-d_n)^n & \longrightarrow & \bigwedge^1 \mathrm{S}'(t-d_n)^n & \longrightarrow & \langle \mathrm{Mon}(t-d^n)\rangle & \longrightarrow & 0 \\
 & \oplus & \searrow^{f_n} & \oplus & \searrow^{f_n} & \oplus & \searrow^{f_n} & \oplus \\
\cdots & \bigwedge^3 \mathrm{S}'(t)^n & \longrightarrow & \bigwedge^2 \mathrm{S}'(t-d_n)^n & \longrightarrow & \bigwedge^1 \mathrm{S}'(t)^n & \longrightarrow & \langle \mathrm{Mon}(t)\rangle
\end{array}$$

where

$$\bigwedge^1 \mathrm{S}'(t)^n := \bigoplus_{j=1}^{n-1}\langle \mathrm{Mon}(t-d_j)\rangle \oplus \langle \mathrm{Mon}(t-\eta')\rangle$$

and for $j > 1$ $\ \bigwedge^j \mathrm{S}'(t)^n$ is defined similarly. It is easy to see that the mapping cone of ψ_{f_n} is the Koszul complex $K^\bullet((f,x_n^{\eta'}),t)$ of the $n+1$ polynomials $(f_1,\ldots,f_n,x_n^{\eta'})$. Thus we have the following long exact sequence of $\mathbf{k}$-spaces:

$$0 \longrightarrow \mathfrak{p}_i\ H^1(K^\bullet((f,x_n^{\eta'}),t)) \longrightarrow \mathfrak{p}_i\ H^0(K^\bullet(f',t-d_n)) \xrightarrow{\cdot f_n} \mathfrak{p}_i H^0(K^\bullet(f',t)) \longrightarrow$$
$$\longrightarrow \mathfrak{p}_i\ H^0(K^\bullet((f,x_n^{\eta'}),t)) \to 0.$$

By the assumption on the genericity of $f_1,\ldots f_{n-1}$ we have that

$$\mathfrak{p}_i\ H^0(K^\bullet(f',t)) \cong \mathfrak{p}_i\langle H_{d'}(t)\rangle$$

where $H_{d'}(t) = \{x^\alpha \mid |\alpha| = t, \alpha_j < d_j \ \forall j < n, \alpha_n < \eta'\}$ of cardinality $\mathcal{H}_{d'}(t)$. Define

$$\delta' := \sum_{j=1}^{n-1} d_j + \eta' = \delta - d_n + \eta' = \nu - d_n + \eta.$$

using that $\eta' = \eta - \delta + \nu$. Then $\mathcal{H}_{d'}(t) = \mathcal{H}_{d'}(\delta' - t)$, therefore

$$\mathcal{H}_{d'}(\nu) = \mathcal{H}_{d'}(\eta - d_n) \ \text{ and } \ \mathcal{H}_{d'}(\nu - d_n) = \mathcal{H}_{d'}(\eta),$$

which implies that

$$\begin{aligned} &\mathfrak{p}_i \ H^0(K^\bullet(f', \eta - d_n)) \cong \mathfrak{p}_i \ H^0(K^\bullet(f', \nu)) \quad \text{and} \\ &\mathfrak{p}_i \ H^0(K^\bullet(f', \nu - d_n)) \cong \mathfrak{p}_i \ H^0(K^\bullet(f', \eta)). \end{aligned}$$

Moreover, it is easy to see that the map

$$\mathfrak{p}_i \ H^0(K^\bullet(f', \eta - d_n)) \longrightarrow \mathfrak{p}_i \ H^0(K^\bullet(f', \eta))$$

induced by the multiplication by f_n is equal to the dual of the map

$$\mathfrak{p}_i \ H^0(K^\bullet(f', \nu - d_n)) \longrightarrow \mathfrak{p}_i \ H^0(K^\bullet(f', \nu))$$

also induced by the multiplication by f_n, i.e. we can find bases such that the matrices of the two maps are transposes of each other. This implies that the complex

$$\begin{aligned} 0 \to \mathfrak{p}_i \ H^1(K^\bullet((f, x_n^{\eta'}), \eta)) \to \mathfrak{p}_i \ H^0(K^\bullet(f', \eta - d_n)) \to \mathfrak{p}_i H^0(K^\bullet(f', \eta)) \to \\ \to \ \mathfrak{p}_i \ H^0(K^\bullet((f, x_n^{\eta'}), \eta)) \to 0. \end{aligned}$$

is the same as the complex

$$\begin{aligned} 0 \to \mathfrak{p}_i \ H^0(K^\bullet((f, x_n^{\eta'}), \nu))^* \to \mathfrak{p}_i \ H^0(K^\bullet(f', \nu))^* \to \mathfrak{p}_i H^0(K^\bullet(f', \nu - d_n))^* \to \\ \to \mathfrak{p}_i \ H^1(K^\bullet((f, x_n^{\eta'}), \nu))^* \to 0. \end{aligned}$$

By our assumption (52) we have that

$$\dim_{\mathbf{k}} \ \mathfrak{p}_i \ H^1(K^\bullet((f, x_n^{\eta'}), \eta)) = |C_1| - l,$$

therefore

$$\dim_{\mathbf{k}} \ \mathfrak{p}_i \ H^0(K^\bullet((f, x_n^{\eta'}), \nu)) = |C_1| - l$$

as we claimed in (53). This concludes the proof of Lemma 3.3.12. ■

Acknowledgments: I would like to thank Elizabeth Mansfield for her continued support and helpful suggestions. I am grateful to Lajos Rónyai to draw my attention to the article [15] used in the proof of Proposition 3.1.6.

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